MATH 472

INTRODUCTION TO ANALYSIS II

SESSION no. 32

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Multiple Integration (Double)

$$f : D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R} \qquad D = [a,b] \times [c,d]$$

$$P_1 = \{a = x_0 < x_1 < x_2 < \cdots < x_n = b\}$$

$$P_2 = \{c = y_0 < y_1 < y_2 < \cdots < y_m = d\}$$

$$P = P_1 \times P_2 = \{R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]\}_{\substack{1 \le i \le n \\ 1 \le j \le m}}$$

$$m_{ij} = \inf\{f(x,y) : x,y \in R_{ij}\}$$

$$M_{ij} = \sup\{f(x,y) : x,y \in R_{ij}\}$$

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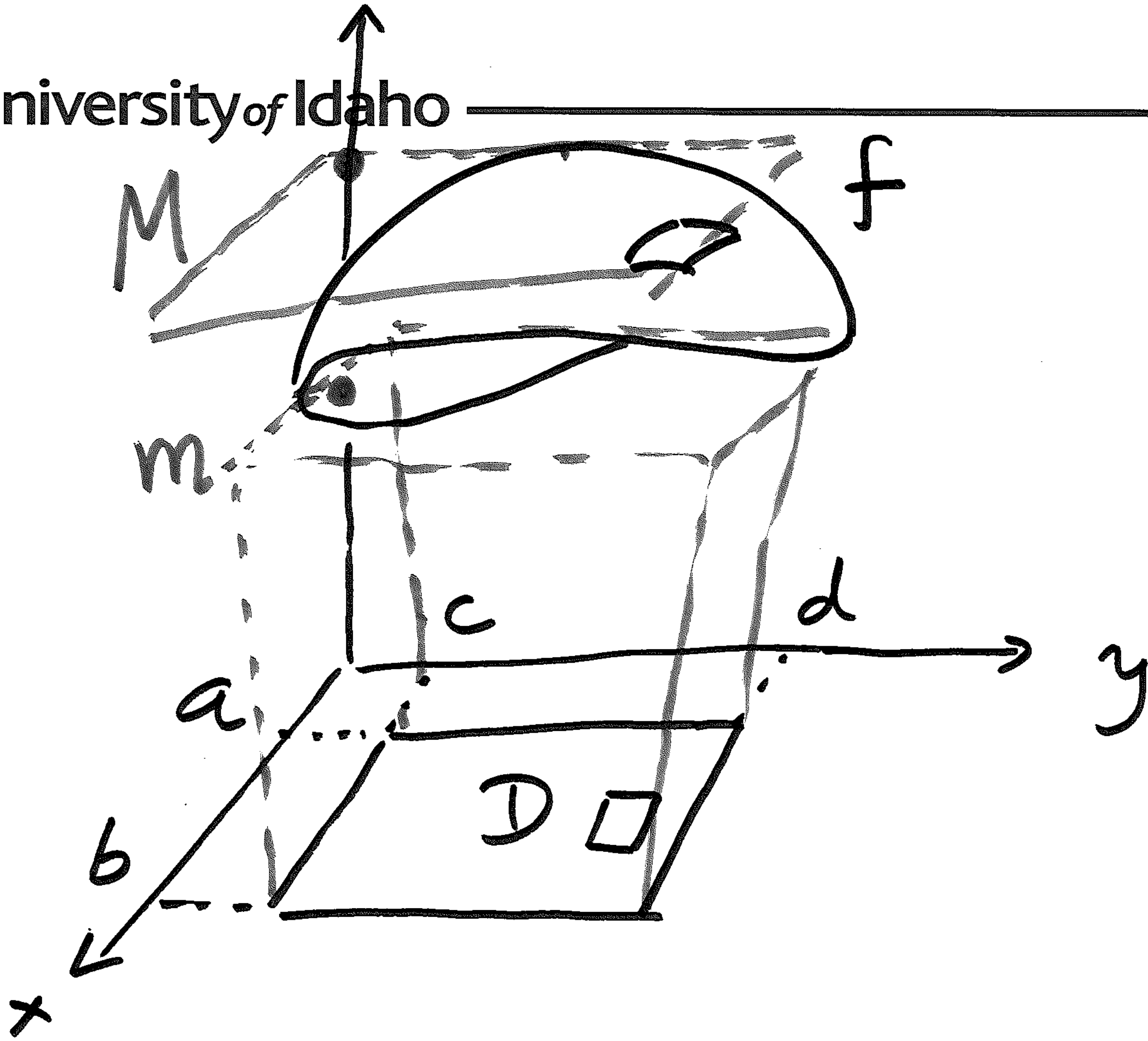

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$$L(P,f) = \sum_{i,j} \overbrace{m_{ij}\, \text{Area}(R_{ij})}^{\text{volume of a cuboid}}$$

$$U(P,f) = \sum_{i,j} \underbrace{M_{ij}\, \text{Area}(R_{ij})}_{\text{volume of a cuboid}}$$

Lemma

$f : D \longrightarrow \mathbb{R}$, $D = [a,b] \times [c,d]$ $\subseteq \mathbb{R}^2$

P is a partition of D

If

$$m \le f(x,y) \le M \quad \forall (x,y) \in D$$

then

$$\underbrace{m \overbrace{(b-a)(d-c)}^{\text{area of } D}}_{\substack{\text{Volume of the} \\ \text{cuboid of height} \\ m \text{ \& base } D}} \le L(P,f) \le U(P,f) \le M \overbrace{(b-a)(d-c)}^{\text{area of } D}$$

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Proof

$$m_{ij} = \inf_{(x,y)\in R_{ij}} f(x,y), \quad M_{ij} = \sup_{(x,y)\in R_{ij}} f(x,y)$$

$$m \le m_{ij} \le M_{ij} \le M$$

$$\Rightarrow m\,A(R_{ij}) \le m_{ij}A(R_{ij}) \le M_{ij}A(R_{ij}) \le M A(R_{ij})$$

$$\Rightarrow m\underbrace{\sum_{i,j} A(R_{ij})}_{\text{area of } D} \le \sum_{i,j} m_{ij}A(R_{ij}) \le \sum_{i,j} M_{ij}A(R_{ij})$$

$$\le M\sum_{i,j} A(R_{ij})$$

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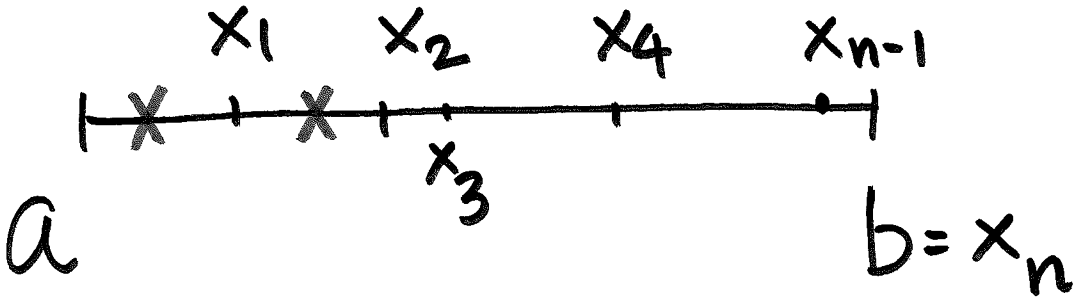
$$\Rightarrow m(b-a)(d-c) \le L(P,f) \le U(P,f)$$
$$\le M(b-a)(d-c)$$

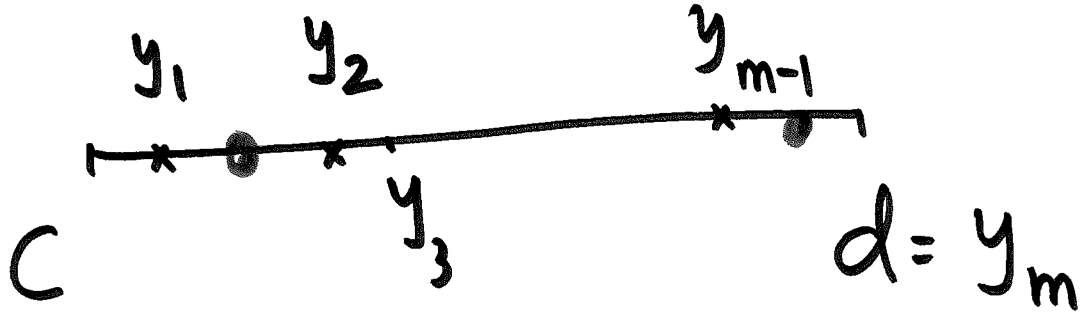
$\square$

P_1



Q_1 : refinement of P_1

P_2



Q_2 : refinement of P_2

Q_1 (Q_2) contains all points of P_1 (P_2) and some extra points.
(finite no. of)

$Q \supseteq P$

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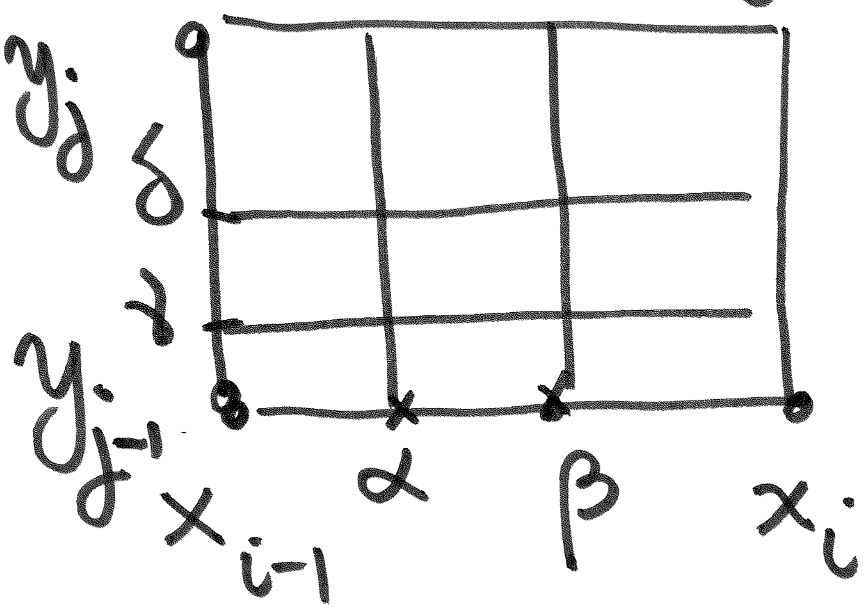
Then $Q = Q_1 \times Q_2$ is a refinement of $P = P_1 \times P_2$

R_{ij} is a rectangle in P



R_{ij}

Add new points $\alpha, \beta, \gamma, \delta$

9\. Refinement Lemma

$f: D \to \mathbb{R}$ is a bounded function. P is a partition of D; Q is a refinement of P. Then

$$L(f,P) \le L(f,Q) \le U(f,Q) \le U(f,P)$$

Proof: R_{ij} is a rectangle in P and $Q(R_{ij})$ is the partition of R_{ij} induced by the refinement Q.

By the previous lemma:

$$m_{ij}\,\text{area}(R_{ij}) \le L(f, Q(R_{ij})) \le U(f, Q(R_{ij})) \le M_{ij}\,\text{area}(R_{ij})$$

[use R_{ij} instead of D]

Add over all i, j to get

$$L(f,P) \le L(f,Q) \le U(f,Q) \le U(f,P). \quad \square$$

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Proposition

For any two partitions P and Q of

1 :

$$L(f, P) \le U(f, Q).$$

Proof: P^* be a common refinement of P and Q. [Eg. $P^* = P \cup Q$].

Then

$$\underbrace{L(f, P) \le L(f, P^*)}_{\text{refinement lemma}} \le U(P^*, f) \overset{\text{refinement lemma}}{\le} U(P, f)$$

□

Upper and Lower Integrals

$f: D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a bounded function

Lower integral

$$\underline{\int_D} f = \sup \{ L(f,P) : P \text{ is a partition of } D \}$$

Upper integral

$$\overline{\int_D} f = \inf \{ U(f,P) : P \text{ is a partition of } D \}$$