MATH 472

INTRODUCTION TO ANALYSIS II

SESSION no. 33

Upper & Lower Integrals

$$\underline{\int_D} f = \sup_P L(f,P)$$

$$\overline{\int_D} f = \inf_P U(f,P)$$

Lemma: $\underline{\int_D} f \le \overline{\int_D} f$

Proof: For any partition P

$$L(P,f) \le U(P,f)$$

$$\underline{\int}_D = \sup_P L(P,f) \le U(P,f)$$

$$L(P,f) \le \inf_P U(P,f) = \overline{\int}_D$$

$$\Longrightarrow \underline{\int}_D f \le \overline{\int}_D f$$

$\square$

Definition

$f : D \longrightarrow \mathbb{R}$, f is bounded.

Then f is said to be integrable if

$$\underline{\int}_D f = \overline{\int}_D f$$

The integral of f, denoted by $\int_D f$, is the common value of the lower & upper integral

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Examples

1) $f(x,y) = K$ for all $(x,y) \in D$

$D = [a,b] \times [c,d]$

For any partition P, for each R_{ij},

$$m_{ij} = \inf_{(x,y)\in R_{ij}} f = K, \quad M_{ij} = K$$

$$U(P,f) = \sum_{i,j} M_{ij}\, a(R_{ij}) = K \sum_{i,j} a(R_{ij})$$

$$= K(b-a)(d-c)$$

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$$\left.\begin{aligned} \overline{\int_D} f &= K(b-a)(d-c) \\ L(P,f) &= K(b-a)(d-c) \\ \underline{\int_D} f &= K(b-a)(d-c) \end{aligned}\right\} \text{same}$$

Thus, f is integrable and

$$\int_D f = K\underbrace{(b-a)(d-c)}_{\text{volume of the space bounded by the graph of } f}$$

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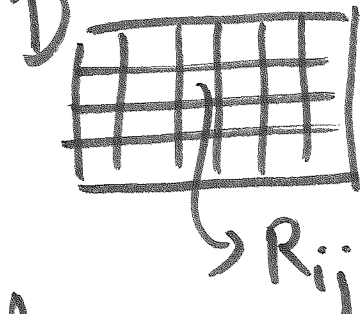
Example 2.

$$f(x,y) = \begin{cases} 1 & \text{if } y \text{ is irrational} \\ 0 & \text{if } y \text{ is rational} \end{cases}$$

$D = [a,b] \times [c,d]$

For any partition P,



$m_{ij} = 0 \qquad M_{ij} = 1$

Any R_{ij} has a point with irrational y co-ordinate and a point with rational y coordinate

$L(P,f) = 0$

$U(P,f) = (d-c)(b-a)$

independent of P

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$$\underline{\int_D} f = 0 \neq \overline{\int_D} f = \underbrace{(b-a)(d-c)}_{\text{area of } D}.$$

$\Rightarrow$ f is not integrable.

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Archimedes-Riemann Theorem

Suppose that $\{P_n\}_{n=1}^{\infty}$ is a sequence of partitions of D such that

$$\lim_{n\to\infty} L(P_n, f) = \lim_{n\to\infty} U(P_n, f).$$

Then f is integrable and

$$\int_D f = \lim_{n\to\infty} L(P_n, f) = \lim_{n\to\infty} U(P_n, f)$$

The converse is also true.

Example

$D = [0,1] \times [0,1]$, $f(x,y) = x^2 y^2$

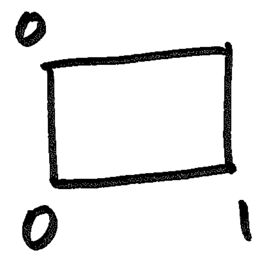
$\text{area}(R_{ij}) = \frac{1}{n^2}$

Let

$$P_n = \left\{ \overbrace{\left[\frac{i-1}{n}, \frac{i}{n}\right] \times \left[\frac{j-1}{n}, \frac{j}{n}\right]}^{R_{ij}},\ 1 \le i, j \le n \right\}$$

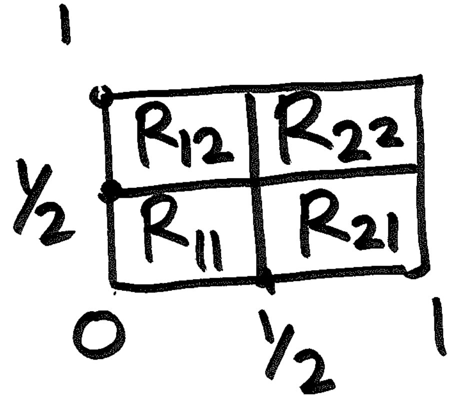
$\{P_n\}_{n=1}^{\infty}$ is a sequence of partitions.

$n = 1$ $\quad P_1 = D$



$n = 2$ $\quad 0, \frac{1}{2}, 1$ along x

$\quad 0, \frac{1}{2}, 1$ along y



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$$m_{ij} = \inf\{f(x,y) : (x,y) \in R_{ij}\}$$

$$= \frac{(i-1)^2 (j-1)^2}{n^4}$$

$$M_{ij} = i^2 j^2 / n^4 \; ; \; U(P_n, f) = \sum_{i,j=1}^{n} \frac{i^2 j^2}{n^4} \underbrace{\frac{1}{n^2}}_{\text{area}(R_{ij})}$$

$$U(P_n, f) = \frac{1}{n^6} \sum_{i=1}^{n} i^2 \sum_{j=1}^{n} j^2$$

$$= \frac{1}{n^6} \left[\frac{n(n+1)(2n+1)}{6}\right]^2$$

$$= \frac{1}{36n^6}(n^4+2n^3+n^2)(4n^2+4n+1)$$

$$= \frac{4n^6}{36n^6} + \underbrace{C_1\frac{n^5}{n^6} + C_2\frac{n^4}{n^6} \cdots}_{\xrightarrow[n\to\infty]{} 0}$$

$$\longrightarrow \frac{1}{9}$$

$$U(P_n,f) \longrightarrow \frac{1}{9}, \quad n \to \infty$$

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$$L(P_n, f) = \sum m_{ij}\, \text{area}(R_{ij})$$

$$= \sum_{i,j=1}^{n} \frac{(i-1)^2 (j-1)^2}{n^4} \frac{1}{n^2}$$

$$= \frac{1}{n^6} \sum_{i=1}^{n} (i-1)^2 \sum_{j=1}^{n} (j-1)^2$$

$$= \frac{1}{n^6} \sum_{i=1}^{n-1} i^2 \sum_{j=1}^{n-1} j^2$$

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$$= \frac{1}{n^6}\left\{\frac{(n-1)n(2n-1)}{6}\right\}^2$$

$$\longrightarrow \frac{1}{9} \text{ as } n \to \infty$$

$$L(P_n, f) \longrightarrow \frac{1}{9}, \quad n \to \infty$$

$$\lim_{n\to\infty} L(P_n, f) = \lim_{n\to\infty} U(P_n, f) = \frac{1}{9}$$

$$\Rightarrow f \text{ is integrable } \boxed{\int_D f = \frac{1}{9}}$$

By Archimedes Riemann Thm.