MATH 472

INTRODUCTION TO ANALYSIS II

SESSION no. 34

1 Recall the Archimedes-Riemann Thm.

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$\{P_n\}_{n=1}^{\infty}$ $f: D \longrightarrow \mathbb{R}$

$$\lim_{n\to\infty} L(P_n, f) = \lim_{n\to\infty} U(P_n, f) = I$$

Given ε, $\exists\ N \in \mathbb{N}$ s.t.

$$|L(P_n, f) - I| < \varepsilon, \quad n \geq N$$

$$|U(P_n, f) - I| < \varepsilon, \quad n \geq N$$

$$|U(P_n, f) - L(P_n, f)| = |U(P_n, f) - I + I - L(P_n, f)|$$

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$$\leq \left| U(P_n, f) - I \right| + \left| L(P_n, f) - I \right|$$

$$\leq 2\varepsilon \quad , \quad n \geq N$$

$$\Rightarrow \left| U(P_n, f) - L(P_n, f) \right| < 2\varepsilon \, , \quad n \geq N$$

$$\Rightarrow \quad U(P_n, f) - L(P_n, f) < 2\varepsilon \, , \; n \geq N$$

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Theorem

$f: D \to \mathbb{R}$ is integrable if and only if for each $\varepsilon > 0$, $\exists$ a partition P of D such that

$$U(P, f) - L(P, f) < \varepsilon.$$

[This is a Corollary to the Archimedes - Riemann Theorem.]

Properties

1) Linearity: If $f_1, f_2 : D \longrightarrow \mathbb{R}$ are integrable and c_1, c_2 are constants

$$\int_D c_1 f_1 + c_2 f_2 = c_1 \int_D f_1 + c_2 \int_D f_2 .$$

2) Additivity: Suppose D is a rectangle such that $D = D_1 \cup D_2$ * where D_1, D_2 are also rectangles. Then f is integrable on D if and only if

* D_1 & D_2 are disjoint except for a common boundary.

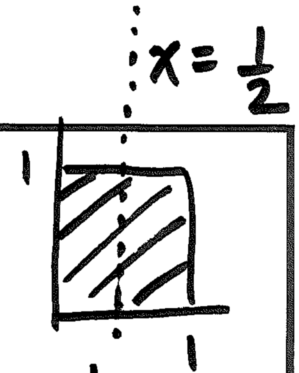
f is integrable on both D_1 & D_2:

$$\int_D f = \int_{D_1} f + \int_{D_2} f$$

3) Monotonicity: $f, g : D \longrightarrow \mathbb{R}$
s.t. $f \leq g$ on D. Then

$$\int_D f \leq \int_D g$$

Example: $D = [0,1] \times [0,1]$



$$f(x,y) = \begin{cases} 5 & \text{if } (x,y) \in D,\ x > \frac{1}{2} \\ 1 & \text{if } (x,y) \in D,\ x \le \frac{1}{2} \end{cases}$$

f is not continuous.

However, f is integrable (by using the Archimedes Riemann Thm]

4) Continuous functions: Any continuous function $f(x,y)$ on D is integrable.

Need to have a technique for evaluating (double) integrals.

Iterated Integrals (Fubini's Theorem)

$f : D \xrightarrow{} \mathbb{R}$ $\quad D \in \mathbb{R}^2$ $\quad D = [a,b] \times [c,d]$

$f(x,y)$

Fix x : f is a function of y only

$f(x,y) = x^2 y \cos y$

$f(x_0, y) = \underbrace{x_0^2}_{c} y \cos y = c\, y \cos y$

a function of just y

Define g as : $g(x) = \int_c^d f(x,y)\, dy$

If g is integrable on $[a,b]$ then

$$\int_a^b \left[\int_c^d f(x,y)\,dy \right] dx \text{ is finite}$$

and is called an interated integral.

Similarly,

$$\int_c^d \left[\int_a^b f(x,y)\,dx \right] dy$$

is also an ~~it~~ iterated integral.

How does this relate to $\int_D f$?

$D = [a,b] \times [c,d]$.

Desire: $\displaystyle\int_D f \overset{?}{=} \int_a^b \int_c^d f(x,y)\,dy\,dx$

$\displaystyle\overset{?}{=} \int_c^d \int_a^b f(x,y)\,dx\,dy$

is this equality always true?

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Fubini's Theorem

$f: D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$ is integrable on $\overbrace{[a,b]\times[c,d]}^{D}$.

If $h(y) \overset{\text{fix } x}{=} f(x,y)$ is integrable in y

and $k(x) \overset{\text{fix } y}{=} f(x,y)$ is integrable in x

then
$$\int_D f = \int_a^b \int_c^d f(x,y)\,dy\,dx$$
$$= \int_c^d \int_a^b f(x,y)\,dx\,dy \, .$$

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Examples (on Fubini)

(1) $f(x,y) = e^{xy}x \qquad D = [1,2]\times[0,1]$

f is continuous and thus satisfies the conditions of Fubini's Thm

$$\int_D f = \int_1^2 \left(\int_0^1 e^{xy} x \, dy \right) dx$$

$$= \int_1^2 \left[\frac{e^{xy}}{x} x \Big|_0^1 \right] dx$$

$$= \int_1^2 \left[e^x - 1\right] dx = e^2 - e - 1$$

2 $D = [0,1] \times [0,1]$

$$f(x,y) = \begin{cases} \dfrac{(x-y)}{(x+y)^3} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

$$\int_0^1 \int_0^1 \frac{(x-y)}{(x+y)^3}\, dx\, dy = -\frac{1}{2}$$

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$$\int_0^1 \int_0^1 \frac{(x-y)}{(x+y)^3}\, dy\, dx = \frac{1}{2}$$

Iterated integrals do not agree. By Fubini's Thm this implies that f is not integrable over D.