MATH 472

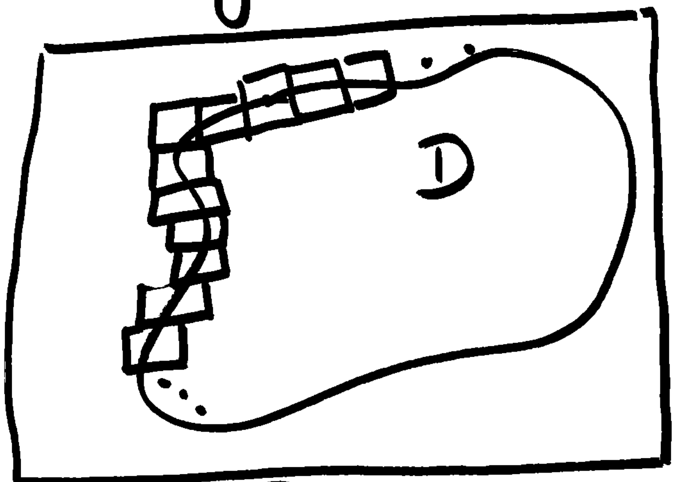
INTRODUCTION TO ANALYSIS II

SESSION no. 35

Integrals over general regions

$f : D \to \mathbb{R}$, D is not necessarily a rectangle

Enclose D in a rectangle P; divide P into smaller rectangles; define a new function



$$F(x,y) = \left.\begin{cases} f(x,y) & (x,y) \in D \\ 0 & (x,y) \notin D \end{cases}\right\} (x,y) \in P$$

$F : P \to \mathbb{R}$, F is called the zero extension of f to P.

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If F is integrable on P then f is integrable on D (since the contribution from points outside D is zero):

$$\int_P F = \int_D f$$

∂D: boundary of D

Suppose that f is continuous on D. Then F must be continuous inside D but may not be continuous on ∂D.

If ∂D is covered by rectangles then the total area of these rectangles $\to 0$ with refinement: Contribution to the integral from ∂D is zero. Therefore if f is continuous on D then the zero extension F is integrable and

$$\int_P F = \int_D f$$

Notation:

$$\iint_D \quad \text{or} \quad \int_D$$

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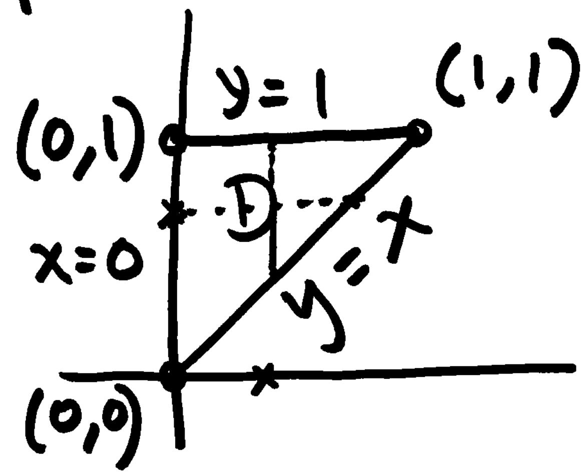
Example

$$\iint_D \sin y^2$$



$\sin y^2$ is continuous $\Rightarrow$ $\sin y^2$ is integrable

$\Rightarrow$ we can apply Fubini's Theorem

$$\int_D \sin y^2 = \int_0^1\int_x^1 \sin y^2 \, dy\, dx = \int_0^1\int_0^y \sin y^2 \, dx\, dy$$

$$= \int_0^1 \left[x \sin y^2\right]_0^y dy = \int_0^1 y \sin y^2 \, dy$$

Substitute $y^2 = u$

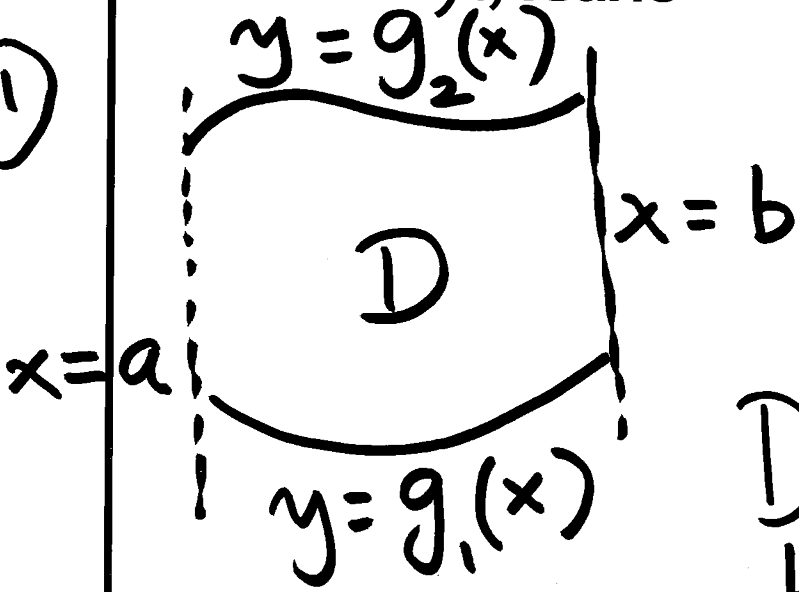
$\vdots$

$= \frac{1}{2}(1 - \cos 1)$

Goal: $\int_D f$ where D is not a rectangle

If Fubini's Thm is applicable then this can be evaluated as an iterated integral.

1

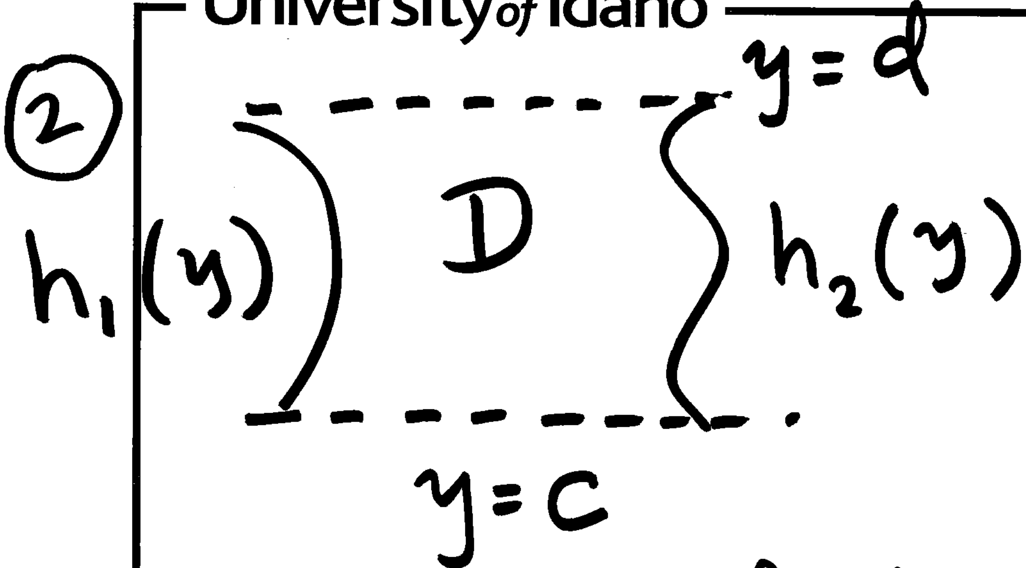


$D = \{(x,y) : g_1(x) \le y \le g_2(x), \; a \le x \le b\}$

Domain of the first type

Then $\displaystyle\int_D f = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\, dy\, dx$

2



$D = \{(x,y) : h_1(y) \le x \le h_2(y),\ c \le y \le d\}$

Domain of the second type

$$\int_D f = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\, dx\, dy$$

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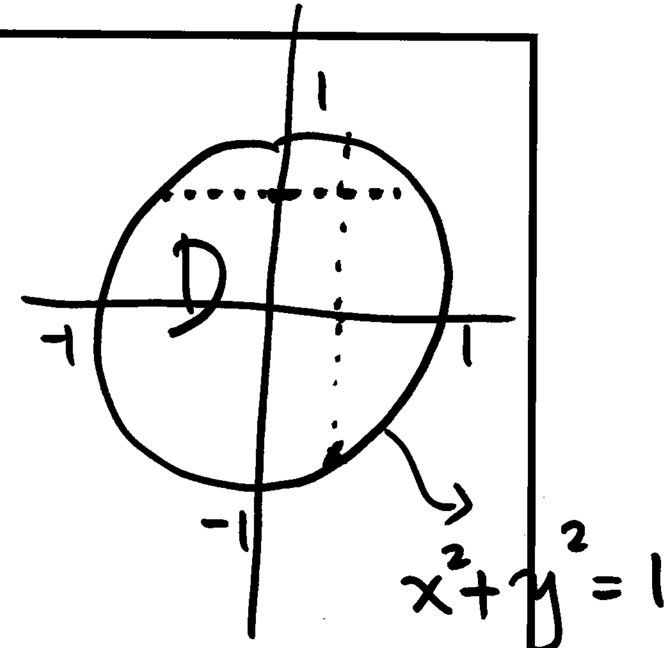
$D = \{(x,y): \ x^2 + y^2 \le 1\}$



As a domain of the first type:

$-1 \le x \le 1, \ -\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$

As a domain of the second type:

$-\sqrt{1-y^2} \le x \le \sqrt{1-y^2}, \ -1 \le y \le 1$

Example

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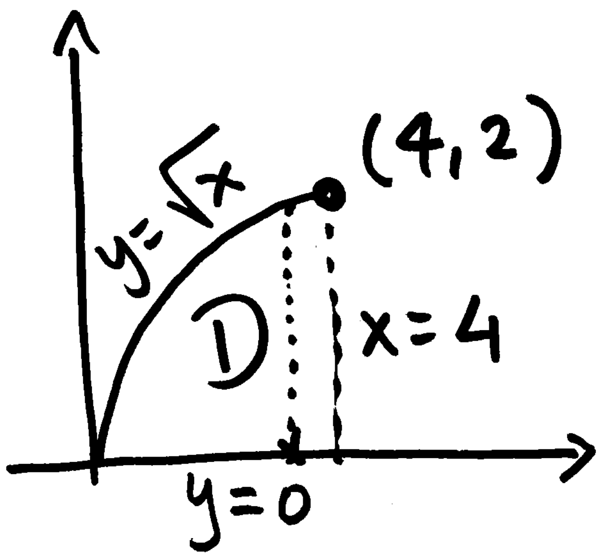
Let $f(x,y)=1$, D is the region between $y=\sqrt{x}$, $y=0$, $x=4$



$$\int_D 1 = \int_0^4 \int_0^{\sqrt{x}} dy\,dx = \int_0^4 y\Big|_0^{\sqrt{x}} dx$$

$$= \int_0^4 \sqrt{x}\,dx = \frac{2}{3} x^{3/2}\Big|_0^4 = \frac{2}{3} 4^{3/2} = \frac{16}{3}$$

In this case $\int_D 1 = \frac{16}{3} =$ area of D (because the function equals 1)