MATH 472

INTRODUCTION TO ANALYSIS II

SESSION no. 36

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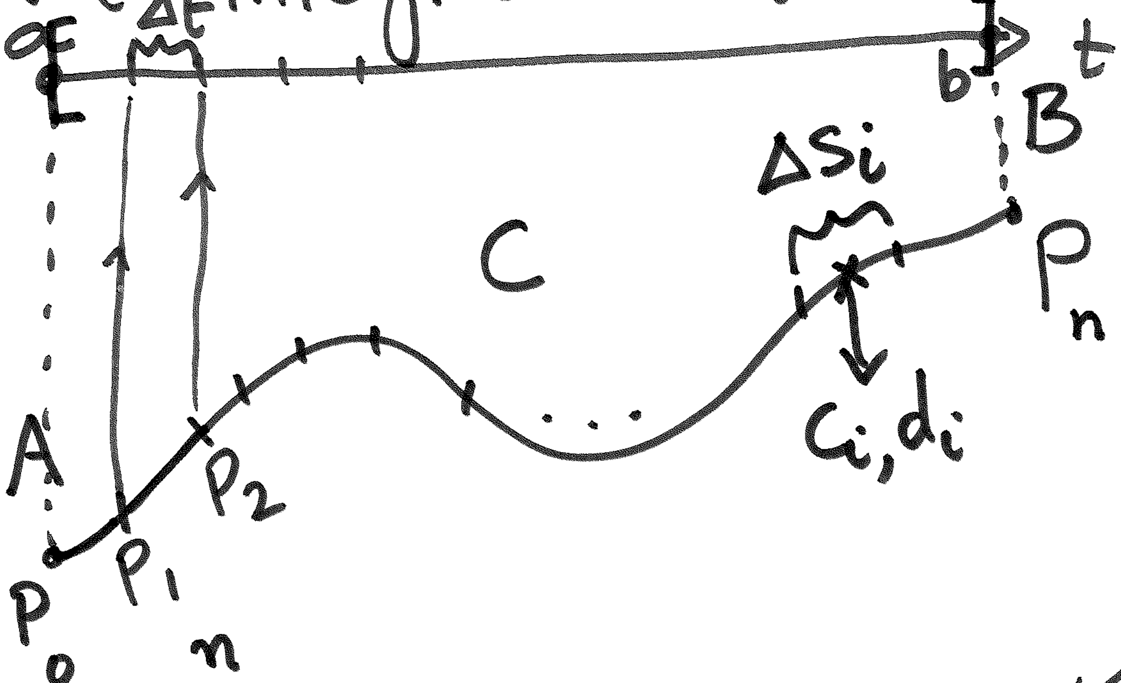
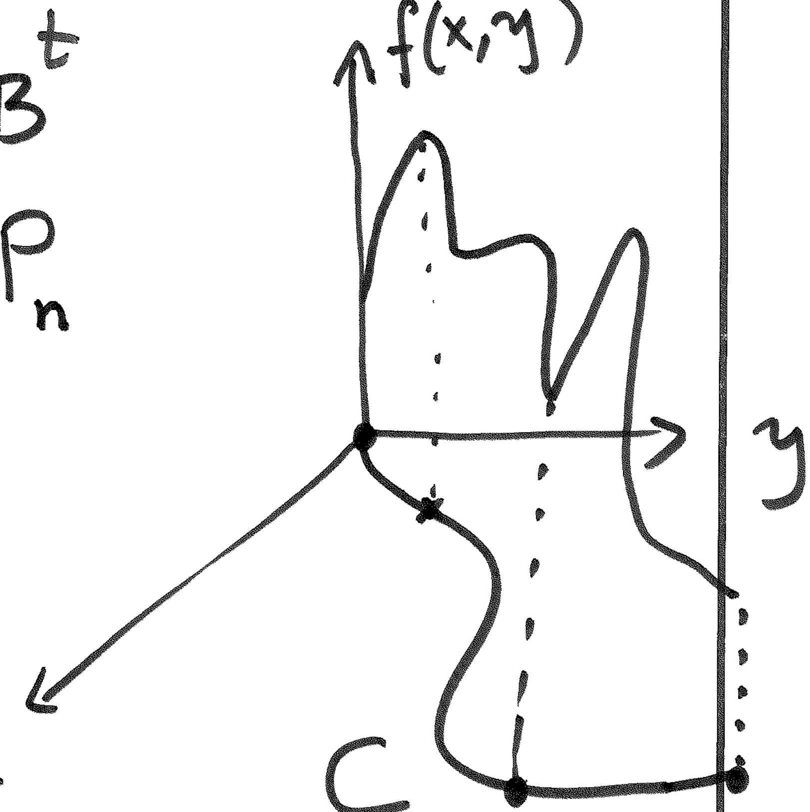
Line Integrals

Instead of integrating on an interval, we integrate on a "smooth" curve





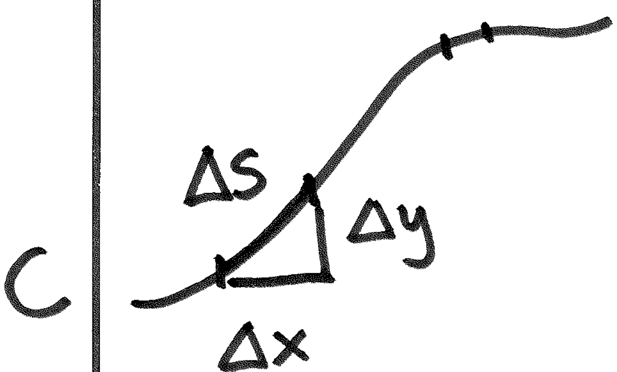
$$L = \lim_{n \to \infty} \sum_{i=1}^{n} f(c_i, d_i)\,\Delta s_i$$

is the line integral of f along C from A to B

$$\int_C f(x,y)\,ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i, d_i)\,\Delta s_i$$

To evaluate: parametrize C:

$x = x(t)$, $y = y(t)$, $a \le t \le b$



$$\Delta s = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

$$\frac{\Delta s}{\Delta t} = \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2}$$

$\Delta t \to 0$, $\Delta s \to 0$, as $n \to \infty$

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$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

which gives

$$\boxed{\int_C f(x,y)\,ds = \int_a^b f(x(t),y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt}$$

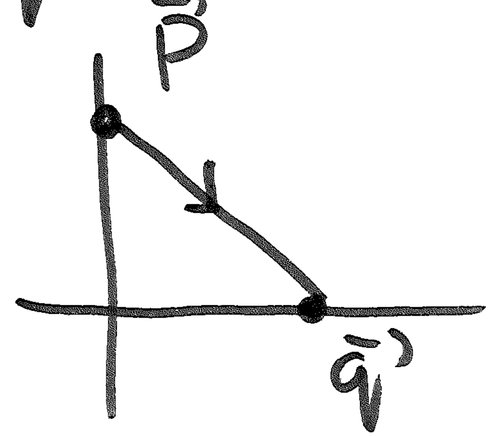
Parametrization of straight lines:

Let $\vec{p}$ & $\vec{q}$ be two points in $\mathbb{R}^2$. Then

$$t\vec{q} + (1-t)\vec{p} \qquad 0 \le t \le 1$$

is a parametrization of the line segment from $\vec{p}$ to $\vec{q}$.

Suppose $\vec{p} = (0,1)$, $\vec{q} = (1,0)$



$$
\begin{aligned}
& t(1,0) + (1-t)(0,1) \\
&= (t,0) + (0, 1-t) \\
&= (t, 1-t)
\end{aligned}
$$

$$
\left.\begin{aligned}
x(t) &= t \\
y(t) &= 1-t
\end{aligned}\right\} \quad 0 \leq t \leq 1
$$

Evaluate $\int_C \overbrace{xy}^{f} \, ds$

C: straight line between $(-1,1), (2,5)$

C: $(1-t)(-1,1) + t(2,5)$

$$\left.\begin{aligned} x(t) &= t-1+2t = 3t-1 \\ y(t) &= 1-t+5t = 1+4t \end{aligned}\right\} 0 \le t \le 1$$

$\frac{dx}{dt} = 3$, $\frac{dy}{dt} = 4$ $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$

$= 5$

$$\int_0^1 (3t-1)(1+4t)\,5\,dt = \frac{25}{2}$$

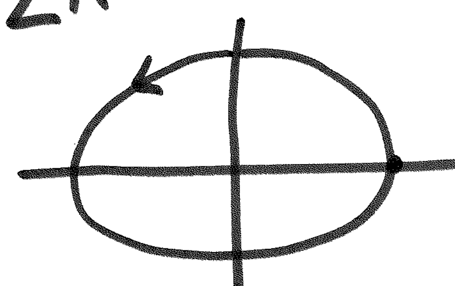
Let C be an ellipse given by

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad a>0,\ b>0$$

$x(t) = a\cos t \qquad 0 \le t \le 2\pi$

$y(t) = b\,\cancel{\cos}\,t$

sin



is a parametrization of the ellipse

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If C is a circle with center O of radius R then

$$x^2 + y^2 = R^2$$

$$C: \begin{array}{l} x(t) = R\cos t \\ y(t) = R\sin t \end{array} \qquad 0 \le t \le 2\pi$$

$$R > 0$$

Integrals to evaluate the work done

$\vec{F} : D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$\vec{F}(x, y) = \big(M(x,y), N(x,y)\big) \in \mathbb{R}^2$

$\vec{F}$ can be thought of as a force field.

In a different context, $\vec{F}$ is called a vector field.

Example

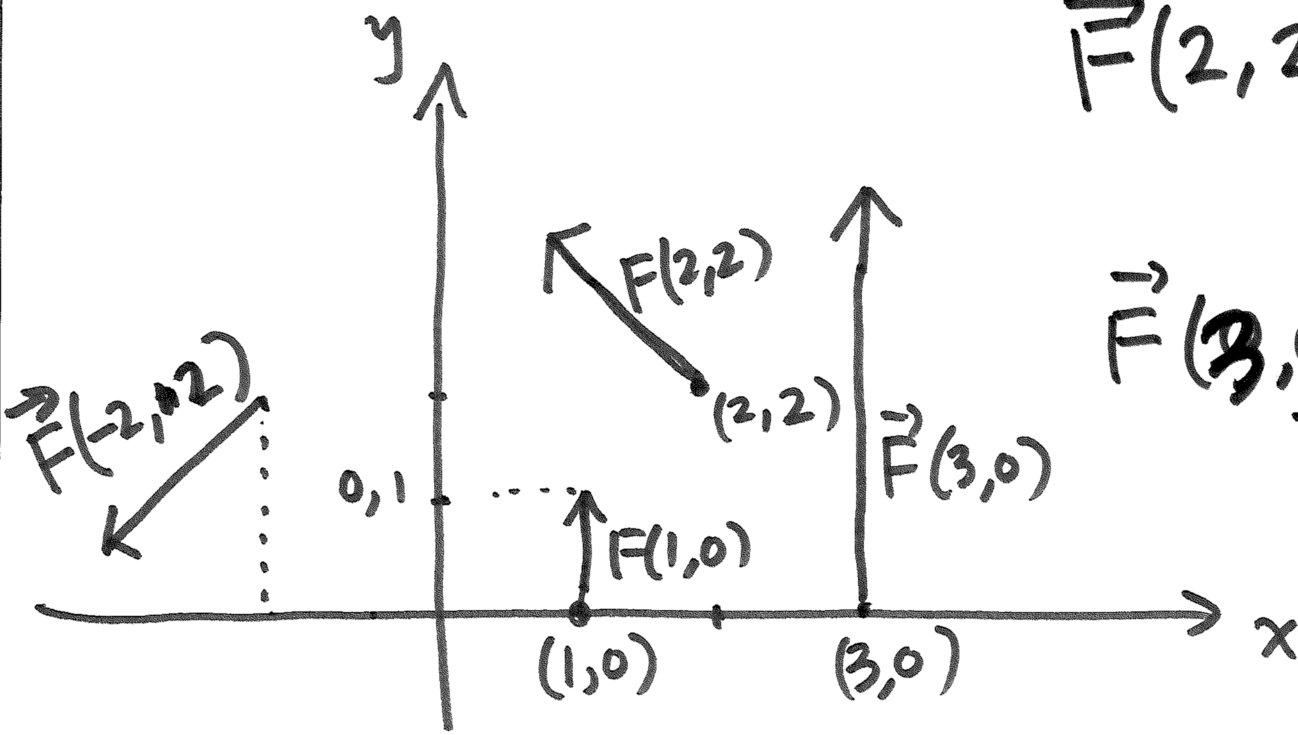
$$\vec{F}(x,y) = (-y, x) \qquad M(x,y) = -y$$
$$N(x,y) = x$$

$$\vec{F}(1,0) = (0,1), \text{ length is } \sqrt{0^2+1^2} = 1$$

$$\vec{F}(2,2) = (-2, 2)$$
$$\sqrt{(-2)^2+2^2} = 2\sqrt{2}$$

$$\vec{F}(3,0) = (0,3)$$

$$\vec{F}(-2,2) = (-2,-2)$$



Consider a curve C given by

$C: (x(t), y(t)) \quad t \in [a, b]$

Goal: Determine the work ~~tin~~ done by a force field $\vec{F}$ along C.

If $\vec{F}$ is a constant force along a straight line:

work done = (force in the direction of the motion) (distance travelled)

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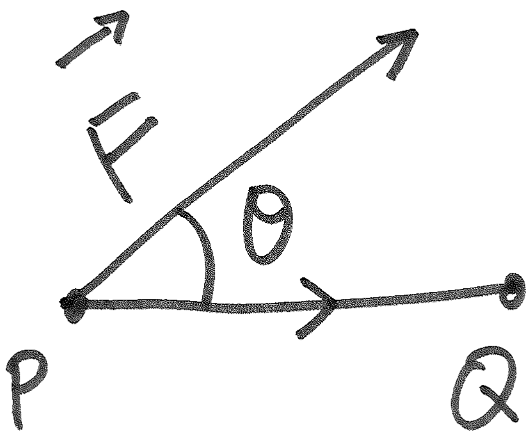

Force along PQ is

$\|\vec{F}\| \cos\theta$

work done =

$\|\vec{F}\| \cos\theta \cdot \underbrace{\text{distance}}_{\|PQ\|}$

$\|\vec{F}\| \, \|PQ\| \cos\theta$