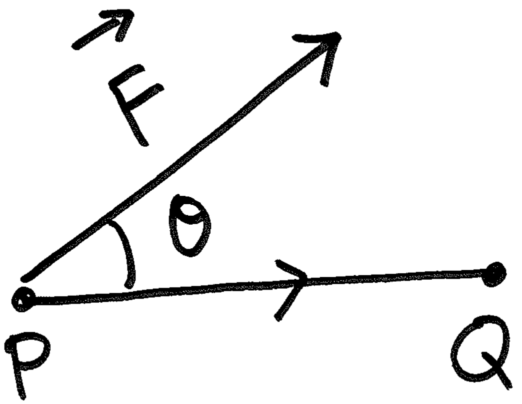
MATH 472

INTRODUCTION TO ANALYSIS II

SESSION no. 37

Work done by $\vec{F}$ along C

If $\vec{F}$ is constant & the motion is along a straight line :



work done = (force along $\vec{PQ}$) (distance moved)

$$= \|\vec{F}\| \cos\theta \, \|\vec{PQ}\|$$

$$= \vec{F} \cdot \vec{PQ}$$

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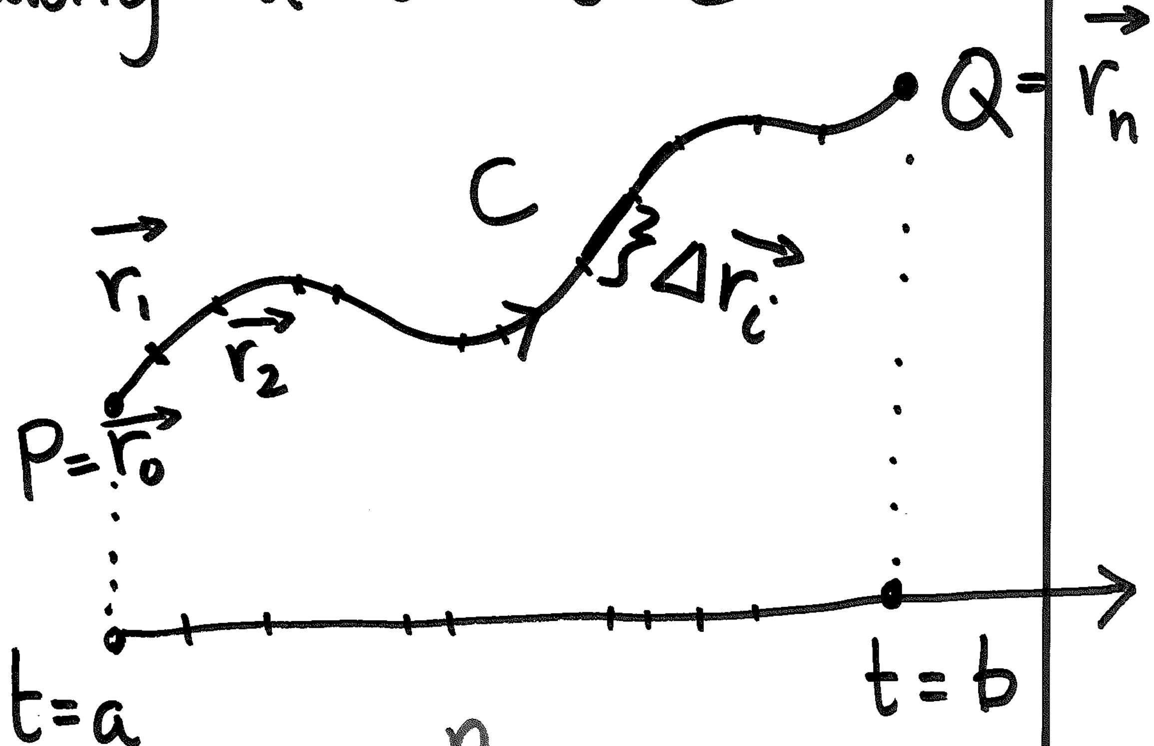
Assume that $\vec{F}$ is not constant and the motion is along a curve C

$$\sum_{i=1}^{n} F(\vec{r}_i)\cdot\Delta\vec{r}_i$$



Take limit $n\to\infty$:

Work done

$$W=\int_C \vec{F}\cdot d\vec{r}=\lim_{n\to\infty}\sum_{i=1}^{n}\vec{F}(\vec{r}_i)\cdot\Delta\vec{r}_i$$

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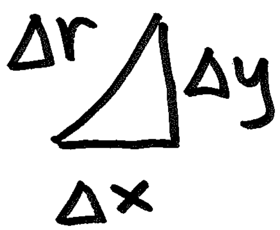

$$\vec{\Delta r} = (\Delta x, \Delta y)$$

$$\vec{F} = (M(x,y), N(x,y))$$

$$\vec{F} \cdot \vec{\Delta r} = M\Delta x + N\Delta y$$

Therefore the work done is

$$\int_C F \cdot dr = \int_C M dx + N dy$$

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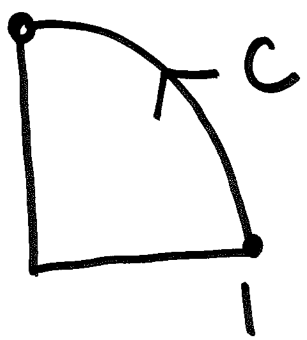
Example

Find the work done by the force $\vec{F}(x,y) = (x^2, -xy)$ in moving along the quarter circle



C: unit circle (first quadrant)

$$C: \begin{cases} x(t) = \cos t, & 0 \le t \le \frac{\pi}{2} \\ y(t) = \sin t \end{cases}$$

$dx = -\sin t\,dt$

$dy = \cos t\,dt$

$$\int_C \vec{F} \cdot dr = \int M\,dx + N\,dy$$

$$= \int x^2\,dx + \int (-xy)\,dy$$

$$= \int_0^{\pi/2} \cos^2 t \, \underbrace{(-\sin t)\, dt}_{dx} + \int -\cos t \sin t \, \underbrace{\cos t \, dt}_{dy}$$

$$= -\int_0^{\pi/2} 2\cos^2 t \sin t \, dt = -\frac{2}{3}$$

Fundamental Thm. of Line Integrals

Recall: Fundamental Theorem of Calculus:

Let F be an integrable function such that $\exists\, f$ s.t. $f' = F$.

Then
$$\int_a^b F(x)\,dx = f(b) - f(a).$$

Gradient: $f : D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right)$$

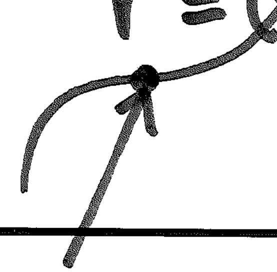
Thm: Fundamental Theorem of line integrals

Let C be a curve in D given by $(x(t), y(t))$ for $t \in [a, b]$. Let $f : D \to \mathbb{R}$ be continuously differentiable.

Then

$$\int_C \nabla f . dr = f(x(b), y(b)) - f(x(a), y(a))$$

$\vec{r} = (x, y)$



Proof

By the chain rule:

$$\frac{d}{dt} f(x(t), y(t)) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

$$= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \cdot \left(\frac{dx}{dt}, \frac{dy}{dt}\right)$$

$$= \nabla f \cdot \left(\frac{dx}{dt}, \frac{dy}{dt}\right)$$

$$\vec{r} = (x, y)$$

$$= \nabla f \cdot \frac{d\vec{r}}{dt}$$

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$$\int_C \nabla f \cdot dr = \int_C \nabla f \cdot \frac{dr}{dt}\, dt$$

$$= \int_a^b \underbrace{\frac{d}{dt}\left[f(x(t), y(t))\right]}_{F(t)} dt$$

FTOC

$$= f(x(b), y(b)) - f(x(a), y(a))$$

$\square$