# MATH 472

# INTRODUCTION TO ANALYSIS II

SESSION no. 38

# Conservative vector fields



$f : D \longrightarrow \mathbb{R}$ Gradient $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$

A vector field $\vec{F} : D \longrightarrow \mathbb{R}^2$ is called a <u>conservative vector field</u> if and only if $\vec{F}$ is the gradient of some $f$ i.e.

$D \subseteq \mathbb{R}^2$

$$\vec{F} = \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$$

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Theorem: If $\vec{F}(x,y) = (M(x,y), N(x,y))$ is continuously differentiable and if $M(x,y) = \frac{\partial f}{\partial x}$ and $N(x,y) = \frac{\partial f}{\partial y}$ for some function $f$ then

$$\frac{\partial M}{\partial y}(x,y) = \frac{\partial N}{\partial x}(x,y).$$

$M_y$

$N_x$

$$\frac{\partial M}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} \underset{\text{by}}{=} \frac{\partial N}{\partial x} = \frac{\partial^2 f}{\partial x \partial y}$$

Clairaut's Theorem

# Conversely:

Theorem: Suppose that $\vec{F} = (M(x,y), N(x,y))$ is continuously differentiable.

If $M_y = N_x$ then $\exists f$ such that $\frac{\partial f}{\partial x} = M$ and $\frac{\partial f}{\partial y} = N$

i.e. $\vec{F}$ is conservative.

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# Examples

1) $\vec{F}(x,y) = (x-y\ ,\ x-2)$

Determine whether $\vec{F}$ is conservative

$M(x,y) = x - y$ $\qquad \dfrac{\partial M}{\partial y} = -1$

$N(x,y) = x - 2$ $\qquad \dfrac{\partial N}{\partial x} = 1$

$\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x} \Rightarrow \vec{F}$ is not conservative

2) $\vec{F}(x,y) = (3 + 2xy,\ x^2 - 3y^2)$

Determine whether $\vec{F}$ is conservative and if so find an $f$ s.t. $\nabla f = \vec{F}$.

$M = 3 + 2xy \qquad M_y = 2x$

$N = x^2 - 3y^2 \qquad N_x = 2x$

$M_y = N_x \Rightarrow \vec{F}$ is conservative

Then find $f$ s.t. $\nabla f = \vec{F}$.

$$\frac{\partial f}{\partial x} = 3 + 2xy \quad \& \quad \frac{\partial f}{\partial y} = x^2 - 3y^2$$

$\Downarrow$

Integrate w.r.t. $x$

$$f = 3x + x^2y + g(y)$$ Find $g$.

Differentiate w.r.t. $y$

$$\frac{\partial f}{\partial y} = x^2 + g'(y)$$

Set $x^2 + g'(y) = x^2 - 3y^2$

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$$\Rightarrow \quad g'(y) = -3y^2$$

$$\Rightarrow \quad g(y) = -y^3 + C$$

$$\Rightarrow \quad f = 3x + x^2y + (-y^3) + C$$

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By the FT of line integrals, if $\vec{F}$ is a conservative vector field then

$$\int_{C_1} \underbrace{\vec{F}}_{\nabla f} \cdot dr = \int_{C_2} \underbrace{\vec{F}}_{\nabla f} \cdot dr$$

where $C_1$ & $C_2$ can be any two different paths with the same start and finish (end-points)

The following are equivalent

(a) $\vec{F}$ is a conservative vector field

(b) The line integral of $\vec{F}$ is path independent

defn.

Fund. Thm. of line ints.

(c) $\exists$ some $f$ s.t. $\nabla f = \vec{F}$.

$f$ is called the potential

$$\int_C \vec{F} \cdot dr = \int_C \nabla f \cdot dr$$ is path independent.

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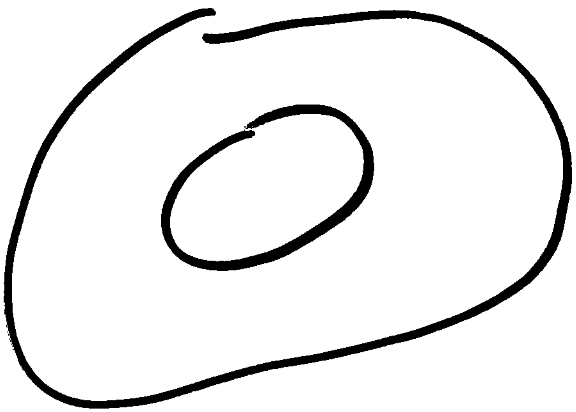
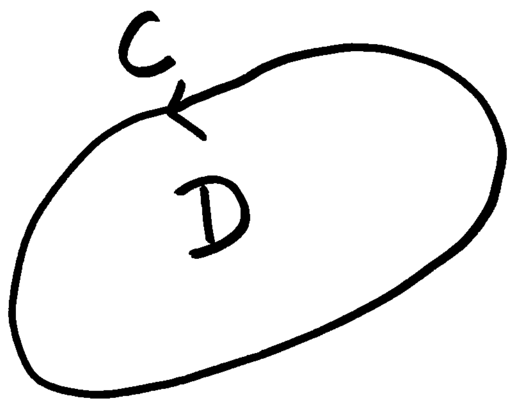
# Green's Theorem

Transformation between double integrals and line integrals.






D - a closed region (include the boundary)

Let C be the boundary of D.

Let $\vec{F}$ be conservative:

$$\int_C \vec{F} \cdot dr = 0$$ by the Fundamental Thm.

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$$\vec{F} = (M(x,y),\ N(x,y))$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad (\vec{F} \text{ is conservative})$$

$$\Rightarrow \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 0$$

$$\Rightarrow \underbrace{\iint_D \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 0 = \int_C \vec{F} \cdot dr}_{\text{Same}}$$

# Green's Theorem

Let $D$ be a closed bounded region whose boundary $C$ consists of finitely many smooth curves. Let $M(x,y)$ & $N(x,y)$ be continuously differentiable. Then

$$\iint_D \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = \int_C M\,dx + N\,dy = \int \vec{F}\cdot dr$$