MATH 472

# INTRODUCTION TO ANALYSIS II

SESSION no. 39

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# Green's Theorem

Let $M(x,y)$ & $N(x,y)$ be continuously differentiable. Then

$$\iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \int_C M\,dx + N\,dy$$

where $C$ is the boundary of the domain $D$, $C$ is traversed such that $D$ is always on the left.

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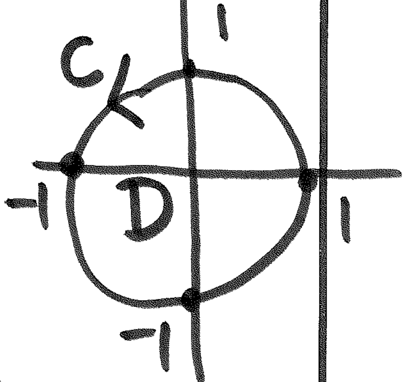
# Verify Green's Theorem



Let $\vec{F} = (\overbrace{y^2 - 7y}^{M}, \overbrace{2xy + 2x}^{N})$

C : boundary of $x^2 + y^2 = 1$




$$\int_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \int_D 2y+2-(2y-7)$$

$$= \int_D 9 = 9\int_D = 9\pi$$

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$0 \leq t \leq 2\pi$

$$\int_C M\,dx + N\,dy$$

Parametrize :

$$\vec{r}(t) = (\underbrace{\cos t}_{x}, \underbrace{\sin t}_{y})$$

$$= \int_0^{2\pi} (\sin^2 t - 7\sin t)\frac{dx}{dt}\,dt + \int_0^{2\pi} (2\cos t\sin t + 2\cos t)\frac{dy}{dt}\,dt$$

$$= \int_0^{2\pi} (\sin^2 t - 7\sin t)(-\sin t)\,dt + \int_0^{2\pi} (2\cos t\sin t + 2\cos t)\cos t\,dt$$

$$= \int_0^{2\pi} (-\sin^3 t + 7\sin^2 t)\,dt + \int_0^{2\pi} 2\cos^2 t\sin t + \int_0^{2\pi} 2\cos^2 t\,dt$$

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$$= 0 + 7\pi + 2\pi = 9\pi$$

[use $\cos 2t = 1 - 2\sin^2 t$

$\cos 2t = \cos^2 t - \sin^2 t$

$\cos 2t = 2\cos^2 t - 1$ ]

# Proof of Green's Theorem

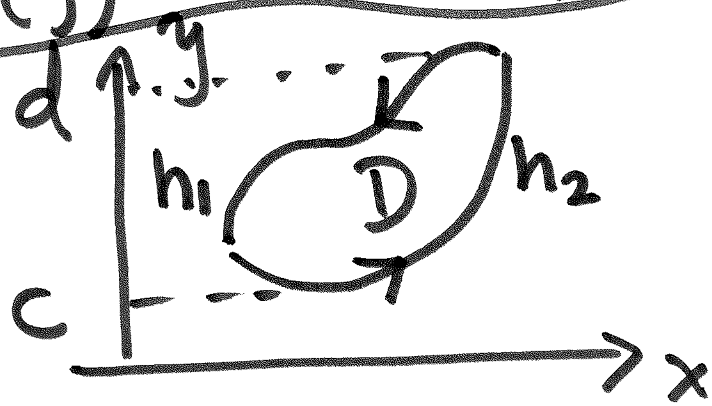
First prove for a "special" region

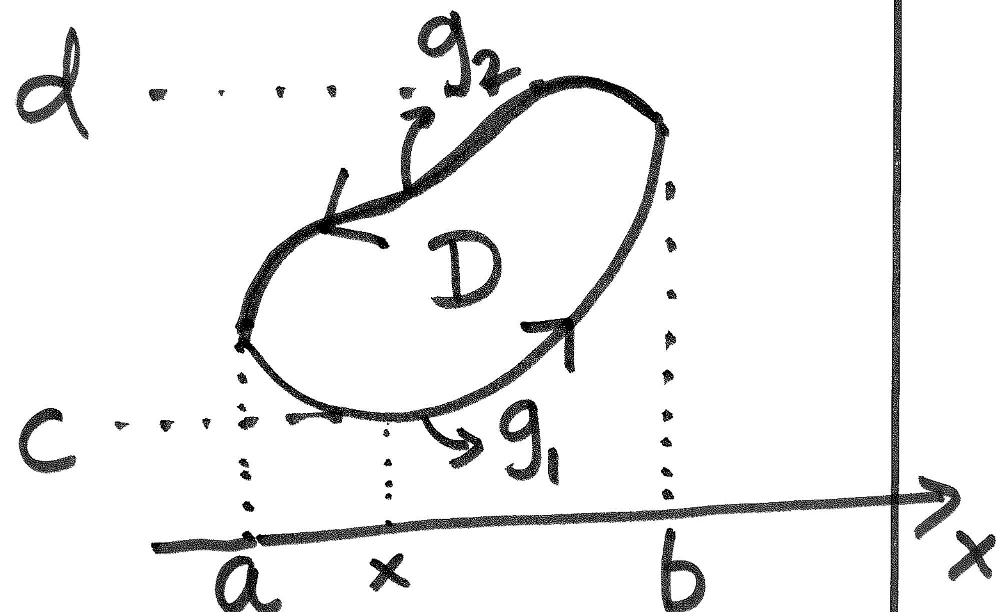
$$a \le x \le b, \quad g_1(x) \le y \le g_2(x)$$

or

$$c \le y \le d$$

$$h_1(y) \le x \le h_2(y)$$



$$\iint_D \frac{\partial M}{\partial y} = \int_a^b \int_{g_1(x)}^{g_2(x)} \frac{\partial M}{\partial y}\, dy\, dx$$

FTOC

$$= \int_a^b \left[ M(x, g_2(x)) - M(x, g_1(x)) \right] dx$$

$$= -\int_b^a M(x, g_2(x))\, dx - \int_a^b M(x, g_1(x))\, dx$$

$$= -\int_C M\, dx$$

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$$\iint_D \frac{\partial N}{\partial x} = \int_c^d \int_{h_1(y)}^{h_2(y)} \frac{\partial N}{\partial x}\, dx\, dy$$

FTOC

$$= \int_c^d \left[ N(h_2(y), y) - N(h_1(y), y) \right] dy$$

$$= \int_c^d N(h_2(y), y)\, dy + \int_d^c N(h_1(y), y)\, dy$$

$$= \int_C N\, dy$$

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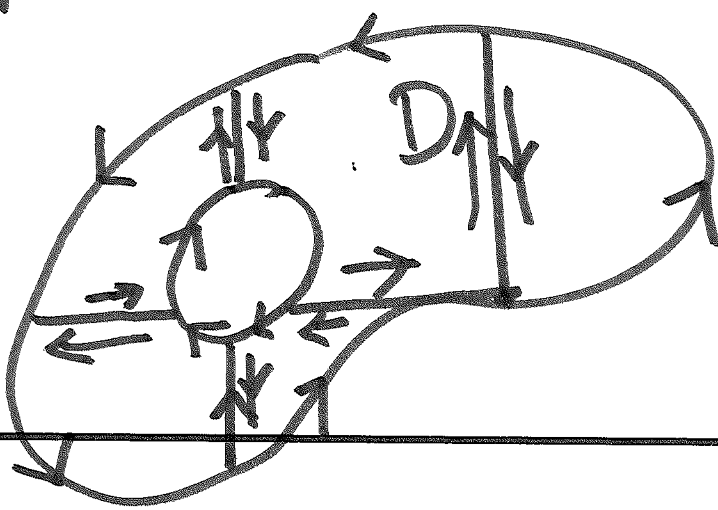
Adding the values for $\int_D \frac{\partial M}{\partial y}$ and $\int_D \frac{\partial N}{\partial x}$:

$$\int_D \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$

$$= \int_C M\,dx + N\,dy$$

If D is of the form:




Subdivide D into regions of the special type and add the results.

The double integrals will add up ~~but~~ while for the line integrals the integrals along the Common boundaries will cancel (due to opposite directions) - We will be left with just the boundary of D.

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Let $\vec{F} = (M(x,y), N(x,y))$

Then $\int_C M\,dx + N\,dy = \int_C \vec{F} \cdot d\vec{r}$

work done

$\vec{r}(t) = (x(t), y(t))$

$d\vec{r} = \cancel{(x'(t), y'(t))}\ (dx, dy) = \left(\frac{dx}{dt}dt, \frac{dy}{dt}dt\right)$

$\vec{F} \cdot d\vec{r} = M\frac{dx}{dt}\,dt + N\frac{dy}{dt}\,dt$

$= M\,dx + N\,dy$

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$C = C_1 \cup C_2$