MATH 472

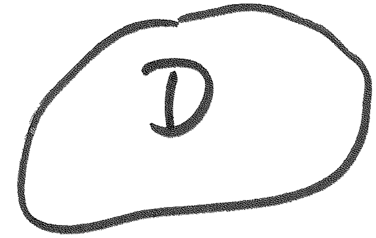
INTRODUCTION TO ANALYSIS II

SESSION no. 40

Finding the area of a region by Green's Thm

Let $M(x,y) = -y$ & $N(x,y) = x$

$\partial M/\partial y = -1$, $\dfrac{\partial N}{\partial x} = 1$



In Green's theorem :

$$\int_D 2 = \int_C -y\,dx + x\,dy$$

$$\Rightarrow \int_D 1. = \frac{1}{2}\int_C -y\,dx + x\,dy$$

$$\Rightarrow \text{area}(D) = \frac{1}{2}\int_C -y\,dx + x\,dy$$

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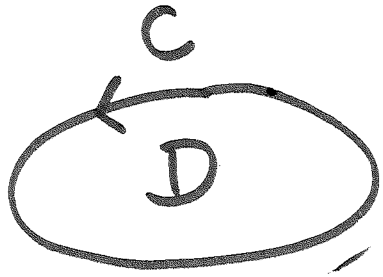
Example

Consider the inside of the ellipse

$$D = \left\{ (x,y) : \frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1 \right\}$$



$x = a\cos\theta \quad dx = -a\sin\theta\, d\theta$

$y = b\sin\theta \quad dy = b\cos\theta\, d\theta$

$0 \le \theta \le 2\pi$

$$\text{Area} = \frac{1}{2}\int_C -y\,dx + x\,dy$$

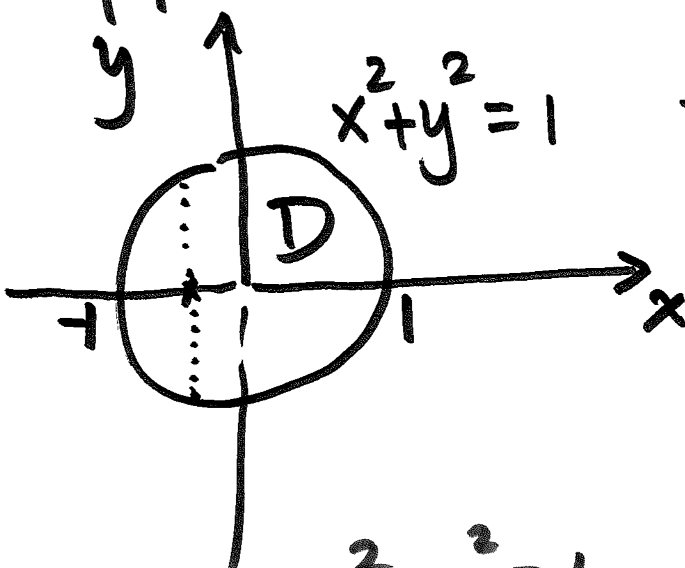
$$= \frac{1}{2}\int_0^{2\pi} -b\sin\theta(-a\sin\theta)d\theta + a\cos\theta\, b\cos\theta\, d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} ab\, d\theta = \pi ab$$

Change of variables in double integrals

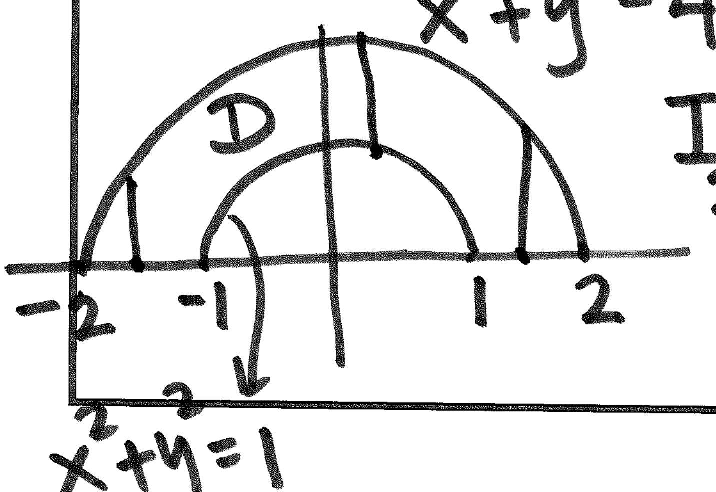
Special case - polar coordinates

Suppose that we have to integrate over



$$I_1 = \iint_D f(x,y) = \int\int f(x,y)\,dy\,dx = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f(x,y)\,dy\,dx$$

too complicated!



$$I_2 = \iint_D f = \int_{-2}^{-1}\int_{0}^{\sqrt{4-x^2}} f(x,y)\,dy\,dx + \int_{-1}^{1}\int_{\sqrt{1-x^2}}^{\sqrt{4-x^2}} f\,dy\,dx + \int_{1}^{2}\int_{0}^{\sqrt{4-x^2}} f$$

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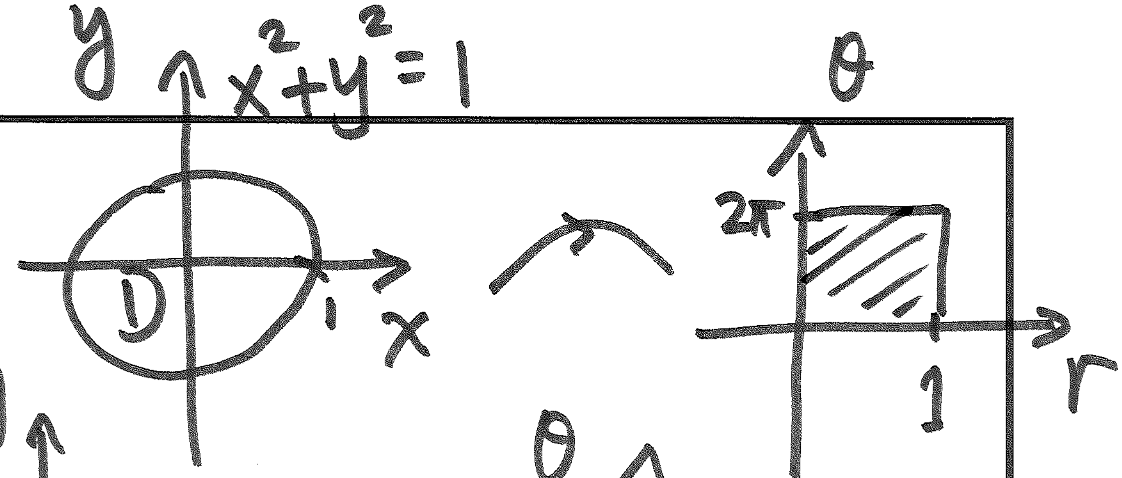
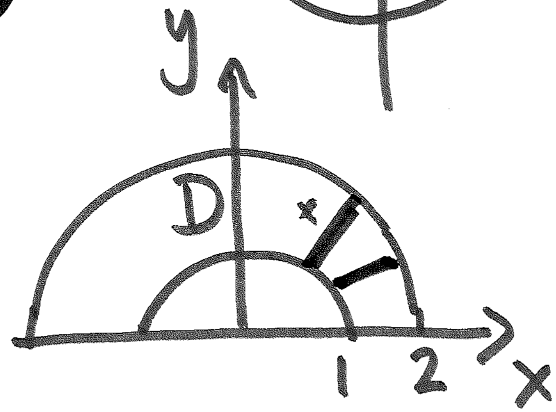
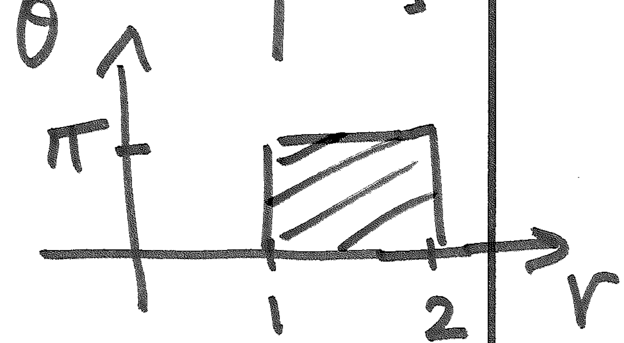
$$x = r\cos\theta$$
$$y = r\sin\theta$$







$$I_1 = \int_0^{2\pi}\int_0^1 f(r,\theta)\,\underbrace{r\,dr\,d\theta}_{dy\,dx}$$

$$I_2 = \int_0^{\pi}\int_1^2 f(r,\theta)\,\underbrace{r\,dr\,d\theta}_{dy\,dx}$$

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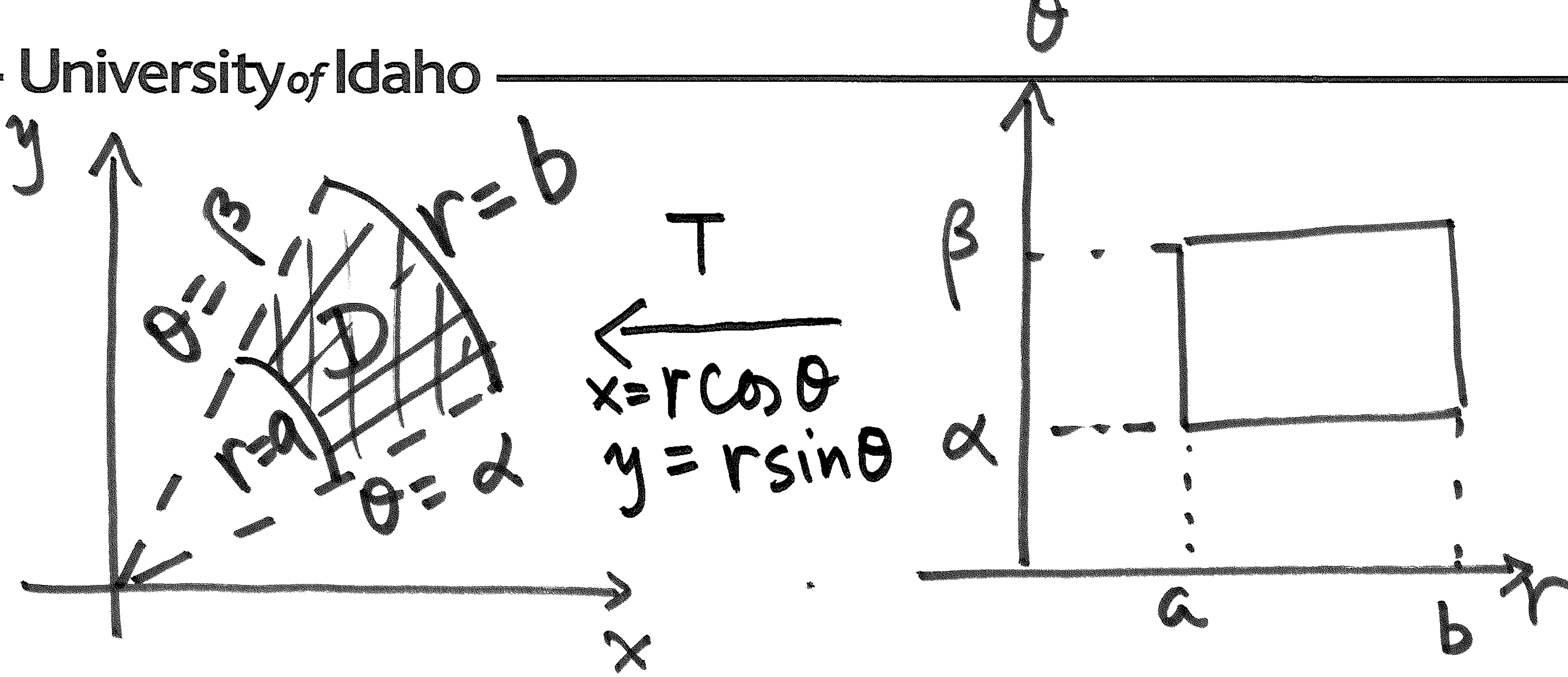
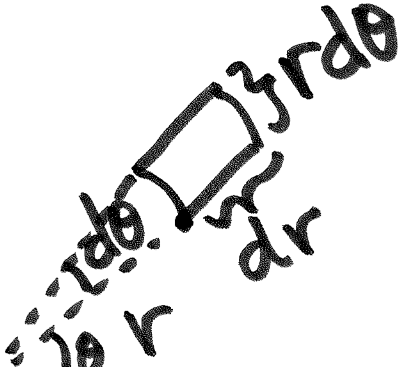


Take a "small" rectangle in the circular domain



$$\text{area} = r\,dr\,d\theta$$
$$= dx\,dy$$

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$$\left.\begin{aligned} x &= r\cos\theta = g(r,\theta) \\ y &= r\sin\theta = h(r,\theta) \end{aligned}\right\} T$$

Jacobian of g, h is

$$J = \begin{bmatrix} \frac{\partial g}{\partial r} & \frac{\partial g}{\partial \theta} \\ \frac{\partial h}{\partial r} & \frac{\partial h}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

$$|J| = r\cos^2\theta + r\sin^2\theta = r$$

$\hookrightarrow$ determinant of J

For polar transformation

$$\int_D f(x,y) = \int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

$$= \int_{y_1}^{y_2}\int_{x_1}^{x_2} f\, dy\, dx$$

$$= \int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2} f(r\cos\theta, r\sin\theta)\, |J|\, dr\, d\theta$$

$$dy\, dx = |J|\, dr\, d\theta$$

Change of variables Theorem

Suppose $f : D \longrightarrow \mathbb{R}$ and T is a one-to-one map from S in the u-v plane to D in the x-y plane given by

$$\left.\begin{aligned} x &= g(u,v) \\ y &= h(u,v) \end{aligned}\right\} T$$

Then

$$\iint_D f(x,y)\, \underbrace{dx\,dy} = \iint_S f(g(u,v), h(u,v))\, |\underbrace{J|\,du}\,dv$$

$$dx\,dy = |J|\,du\,dv$$

$$J = \begin{bmatrix} \partial g/\partial u & \partial g/\partial v \\ \partial h/\partial u & \partial h/\partial v \end{bmatrix}$$