ON WARING’S PROBLEM FOR SYSTEMS OF SKEW-SYMMETRIC FORMS

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ABSTRACT. This paper is devoted to a problem of finding the smallest positive integer $s(m, n, k)$ such that $(m + 1)$ generic skew-symmetric $(k + 1)$-forms in $(n + 1)$ variables as linear combinations of the same $s(m, n, k)$ decomposable skew-symmetric $(k + 1)$-forms.

This problem is analogous to a well known problem called Waring’s problem for symmetric forms and can be very naturally translated into a classical problem in algebraic geometry. In this paper, we will go through some basics of algebraic geometry, describe how objects in algebraic geometry can be associated to systems of skew-symmetric forms, and discuss algebro-geometric approaches to establish the existence of triples $(m, n, k)$, where $s(m, n, k)$ is more than expected.

1. INTRODUCTION

In 1707, E. Waring suggested the problem of expressing every positive integer as a sum of at most $s(d)$ $d^{th}$ powers of positive integers, which was affirmatively solved by Hilbert in 1909.

Recently, there has been tremendous interest in a similar question for systems of algebraic forms, which asks “What is the smallest integer $s(m, n, d)$ such that the $(m + 1)$ generic $d$-form in $(n + 1)$ variables defined over an algebraically closed field of characteristic 0 are expressible as linear combinations of the same $s(m, n, d)$ $d^{th}$ powers of linear forms?” In the case of $m = 0$, this problem is known as a Waring’s problem for algebraic forms or Waring’s problem for polynomials, which had remained unsolved for many years, but was completed by Alexander and Hirschowitz [7] in 1995.

As we shall see in Section 2, we have the inequality $s(m, n, d) \geq \left\lceil \frac{(n+1)(n^d)}{(m+n+1)} \right\rceil$ with equality being what is generally expected. There are examples of triples $(m, n, d)$ such that $s(m, n, d)$ do not have the expected value. We list these exceptions in the table as given below.

The theorem of Alexander and Hirschowitz says that $s(0, n, d)$ must have the expected value unless $(0, n, d)$ falls into one of the first five cases of Table 1. In their recent paper [8], E. Ballico, A. Bernardi, and M. V. Catalisano proved that $(1, 2, 3)$ is the only exception for $m = 1$. Waring’s problem for systems of algebraic forms is still wide open for $m \geq 2$, but considerable efforts have been already made and research in this area has grown significantly in last ten years. In this paper, we change our scope from Waring’s problem for systems of algebraic forms to an analogous problem for systems of skew-symmetric forms. In the following paragraph, we will state this problem explicitly.
\[
\begin{array}{|c|c|c|}
\hline
(m, n, d) & \text{actual } s(m, n, d) & \text{expected } s(m, n, d) \\
\hline
(0, n, 2) & (n + 1) & \left[\frac{n+1}{d}\right]/(n + 1) \\
(0, 4, 3) & 8 & 7 \\
(0, 2, 4) & 6 & 5 \\
(0, 3, 4) & 10 & 9 \\
(0, 4, 4) & 15 & 14 \\
(m, n, d) \text{ with } m \geq \left(\frac{n+1}{d}\right) - n + 1 & \min\left\{m + 1, \left(\frac{n+1}{d}\right)\right\} & \left(\frac{m+1}{d}\right) \cdot \left(\frac{n+1}{m+1}\right) \\
(2, \ell + 1, 2) \text{ with } \ell \in \mathbb{N} & 3\ell + 3 & 3\ell + 1 \\
(4, 3, 2) & 7 & 6 \\
(1, 2, 3) & 6 & 5 \\
\hline
\end{array}
\]

Table 1. Known exceptions of \((m, n, d)\) with unexpected \(s(m, n, d)\)

Let \(k\) be an algebraic closed field of characteristic 0, let \(W\) be an \((n + 1)\)-dimensional vector space over \(k\), and let \(\wedge^{k+1} W\) denote the \((k + 1)^{st}\) exterior power of \(W\). We denote by \(PW\) (or by \(P^n\)) the projective space of lines in \(W\) passing through the origin \(\{0\}\) and by \([w]\) the line containing \(w \in W \setminus \{0\}\). A skew-symmetric tensor \(w\) in \(\wedge^{k+1} W\) is called decomposable if it is of the form \(w_0 \wedge \cdots \wedge w_k\) (or equivalently if \([w]\) \(\in \mathbb{P}(\wedge^{k+1} W)\)) lies in the Grassmannian of \((k + 1)\)-dimensional linear subspaces of \(\mathbb{P} W\), which we denote by \(G(k, PW)\) or by \(G(k, n)\). Note that the decomposable skew-symmetric tensors span \(\wedge^{k+1} W\), and so every skew-symmetric tensor in \(\wedge^{k+1} W\) is expressible as a linear combination of decomposable skew-symmetric tensors. The problem we will discuss here is to find the least integer \(s(m, n, k)\) such that \((m + 1)\) generic skew-symmetric tensors in \(\wedge^{k+1} W\) can be expressed as linear combinations of the same \(s(m, n, k)\) decomposable skew-symmetric tensors. In this paper, we will consider this problem from an algebro-geometric point of view. The following paragraph will be devoted to reformulating Waring’s problem for systems of skew-symmetric forms using the concept of a “Grassmann secant variety.”

Let \(X \subseteq \mathbb{P}^n\) be a projective variety. An \((s - 1)\)-dimensional linear subspace of \(\mathbb{P}^n\) spanned by \(s\) linearly independent points of \(X\) is called a secant \((s - 1)\)-plane to \(X\). The \((m, s)\)-Grassmann secant variety of \(X\) (or just a Grassmann secant variety of \(X\)), denoted \(GS_{m,s}(X)\), is the projective variety obtained as the Zariski closure of the set of \(m\)-dimensional linear subspaces of \(\mathbb{P}^n\) in \(G(m, n)\), each of which is contained in some secant \((s - 1)\)-plane to \(X\). It is worth noticing that \(GS_{0,s}(X)\) is also known as the \(s^{th}\) secant variety of \(X\). As shall been seen in Section 2, the following inequality holds:

\[
\dim GS_{m,s}(X) \leq \min\{s \cdot \dim X + \dim G(m, s), \dim G(m, n)\}
= \min\{s \cdot \dim X + (m + 1)(s - (m + 1)), (m + 1)(n - m)\}.
\]

In general, one expects that Inequality (1.1) should be an equality. If not, then we say that \(X\) is \((m, s)\)-defective (or \(GS_{m,s}(X)\) is defective). If \(m = 0\), then we just say that \(X\) is \(s\)-defective. In the next paragraph, we will see the connection between Waring’s problem for systems of skew-symmetric forms and the “non-defectivity” of Grassmann secant varieties of Grassmann varieties.

Suppose that \((m + 1)\) generic skew-symmetric tensors \(w_0, \ldots, w_m \in \wedge^{k+1} W\) can be written as linear combinations of the same \((m, n, k)\) decomposable skew-symmetric tensors (but not fewer). Then the linear subspace spanned by \([w_0], \ldots, [w_m]\) must lie in a secant \((s - 1)\)-plane to \(G(k, PW)\), and vice versa. Therefore the condition that \(s(m, n, k)\) is the smallest integer such that the \((m + 1)\) generic skew-symmetric tensors in \(\wedge^{k+1} W\) can
be expressed as linear combinations of the same \(s(m, n, k)\) decomposable skew-symmetric tensors is equivalent to the condition that \(GS_{m,(m,n,k)}(\mathbb{G}(k,\mathbb{P}W))\) coincides with \(\mathbb{P} \setminus \ell k+1 W\).

Let \(s \in \mathbb{N}\), let \(d(m, n, k) = s(k + 1)(n - k) + (m + 1)\), and let \(N(m, n, k) = (m + 1)\binom{n+1}{k+1} - (m + 1)\). Then since \(1.1\) leads to the inequality \(s(m, n, k) \geq [N(m, n, d)/d(m, n, k)]\) with equality if and only if inequality \(1.1\) is an equality for \(s = [N(m, n, d)/d(m, n, k)]\), classifying \((m, s)\)-defective Grassmann varieties can be thought of as a natural generalization of Waring’s problem for systems of skew-symmetric forms.

Note that, due to the isomorphism \(\mathbb{G}(k, n) \cong \mathbb{G}(n - k - 1, n)\), we may consider the problem of classifying defective \(GS_{m,s}(\mathbb{G}(k, n))\) only for \(k \leq (n - 1)/2\). If \(m = 0\), then the following cases are known to be defective; namely \(GS_{0,s}(\mathbb{G}(1, n))\) with \(2 \leq s \leq \lfloor(n+1)/2\rfloor\), \(GS_{0,3}(\mathbb{G}(2, 6))\), \(GS_{0,3}(\mathbb{G}(3, 7))\), \(GS_{0,4}(\mathbb{G}(3, 7))\), and \(GS_{0,4}(\mathbb{G}(2, 8))\), and it is believed that there are no further defective Grassmann secant varieties of \(\mathbb{G}(k, n)\) for \(m = 0\). This conjecture was proposed by H. Kaji [17] in 1997 and by K. Bauer, J. Draisma, and W. A. de Graaf [10] in 2009 independently. However, we know of no general conjecture on defective Grassmann secant varieties of Grassmann varieties. Even no examples of defective such varieties, to our best knowledge, have been discussed for \(m \geq 1\) in the literature before.

The main result of this paper is the establishment of previously unknown examples, where Grassmann secant varieties of Grassmann varieties do not have the expected dimension. The defective cases we found are summarized in the following Table:

<table>
<thead>
<tr>
<th>((m, n, k))</th>
<th>(s)</th>
<th>Theorem</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i) ((m, n, k))</td>
<td>(h(m, n, k) &lt; s \leq \min{m, \binom{n+1}{k+1} - 1})</td>
<td>3.4</td>
</tr>
<tr>
<td>(ii) ((2, 4\ell + 2, 1)) with (\ell \in \mathbb{N} \cup {0})</td>
<td>(3\ell + 2)</td>
<td>4.2</td>
</tr>
<tr>
<td>(iii) ((2, 5, 1))</td>
<td>4</td>
<td>5.2</td>
</tr>
<tr>
<td>(iv) ((2, 7, 1))</td>
<td>5</td>
<td>5.2</td>
</tr>
<tr>
<td>(v) ((1, 5, 2))</td>
<td>3</td>
<td>5.2</td>
</tr>
<tr>
<td>(vi) ((2, 5, 2))</td>
<td>5</td>
<td>5.3</td>
</tr>
</tbody>
</table>

Table 2. Defective \(GS_{m,s}(\mathbb{G}(k, n))\)

Here \(h(m, n, k) = \binom{n+1}{k+1} - (k + 1)(n - k)\). We notice that it is an immediate consequence that the value of \(s(m, n, k)\) should be higher than expected for (i), (ii), and (vi).

In order to prove the defectivity of \(GS_{m,s}(\mathbb{G}(k, n))\) for the triples given in Table 2, we use the equivalence between the concept of \(\mathbb{G}(k, n)\) being \((m, s)\)-defective and the concept of the Segre embedding of \(\mathbb{P}^m \times \mathbb{G}(k, n)\) (we call this variety the Segre-Grassmann variety) being \(s\)-defective. This equivalence often makes the problem of determining the defectivity (or non-defectivity) of Grassmann secant varieties more accessible, because the higher secant varieties have already been considerably studied and a wide array of tools including the so-called Terracini lemma to prove or disprove the non-defectivity of such varieties are now available. We use algebraic and geometric approaches based on some of these tools to show that the Segre embedding of \(\mathbb{P}^m \times \mathbb{G}(k, n)\) corresponding to (i)-(vi) are defective. A detailed discussion about these approaches will be given in later sections. Below, for the reader’s convenience, we summarize the actual codimension and the expected codimension of the secant variety of the Segre embedding of \(\mathbb{P}^m \times \mathbb{G}(k, n)\) corresponding to each of (i)-(vi) in Table 3.

This paper is organized as follows: In Section 2 we will recall some basics related to the focus of the paper. Section 3 will be devoted to describing a family of defective Segre-Grassmann varieties called the unbalanced case (see (i) in Table 3). To our best
knowledge, the concept of unbalanced case was first introduced for Segre varieties in [5]. Later, this concept was extended to Segre-Veronese varieties in [12]. In this paper, we will define the concept of unbalanced case for the Segre embedding $\text{Seg}(P^m \times X)$ of the product of projective $m$-space and an arbitrary projective variety $X$, and then we will show that $\text{GS}_0, \text{Seg}(P^m \times X)$ are defective for a certain range of $s$. Section 4 will be devoted to establishing another infinite family of defective Segre-Grassman varieties (see (ii) in Table 3). In Section 5, the defectivity of the remaining sporadic cases, i.e., (iii)-(vi), will be discussed. The proofs for the defectivity of the secant varieties (i)-(vi) suggest that the codimensions of these varieties should be the same as indicated in Table 3. We will show that this is actually the case.

2. Preliminaries

2.1. Notation. Throughout this paper, we always assume that $k$ is an algebraically closed field of characteristic 0.

Let $V$ be a finite-dimensional vector space over $k$. Then we denote by $V^* = \text{Hom}_k(V, k)$ the dual of $V$, by $\text{Sym}_d V$ the $d^{th}$ symmetric power of $V$, and by $\bigwedge^d V$ the $d^{th}$ exterior power of $V$. Let $U$ and $W$ be subspaces of $V$. Then $U + W$ means the vector space sum of $U$ and $W$.

Let $\mathbb{P}(V)$ (or just $\mathbb{P}V$) be the projective space of lines in $V$ passing through the origin $\{0\}$. If $\dim V = n + 1$, then we sometimes write $\mathbb{P}^n$ instead of $\mathbb{P}V$. Let $v \in V \setminus \{0\}$. Then we denote by $[v]$ the equivalence class containing $v$. For a non-empty subset $S$ of $\mathbb{P}V$, let $\langle S \rangle$ be the linear subspace spanned by $S$. If $X$ is a projective variety in $\mathbb{P}V$ and if $p \in X$ is a non-singular point of $X$, then we write $T_pX$ for the projectivized tangent space to $X$ at $p$ and $\mathbb{F}_pX$ for the affine cone over $T_pX$.

2.2. Basic concepts in multilinear algebra. This section contains some basic concepts in multilinear algebra, which will be used in Sections 4 and 5.

2.2.1. Partially symmetric and partially skew-symmetric tensors. Let $V$ and $W$ be finite-dimensional vector spaces over $k$. Then the direct product of the symmetric group $\mathfrak{S}_1$ of order 1 and the symmetric group $\mathfrak{S}_k$ of order $k$ acts naturally on the tensor product of $V$ and $W^\otimes k + 1$. The elements of $V \otimes \bigwedge^{k+1} W \subset V \otimes W^\otimes k + 1$ are partially symmetric and partially skew-symmetric with respect to the action of $\mathfrak{S}_1 \times \mathfrak{S}_{k+1}$.

A non-zero tensor $t \in V \otimes \bigwedge^{k+1} W$ is said to be decomposable if there exist $v \in V$ and $w_0, \ldots, w_k \in W$ such that $t = v \otimes w_0 \wedge \cdots \wedge w_k$. Note that $V \otimes \bigwedge^{k+1} W$ is generated by decomposable partially symmetric and partially skew-symmetric tensors, or equivalently, every tensor in $V \otimes \bigwedge^{k+1} W$ can be written as a linear combination of decomposable tensors in $V \otimes \bigwedge^{k+1} W$. We say that $t \in V \otimes \bigwedge^{k+1} W$ is said to have rank $s$ if $t$ is expressible as a
linear combination of \( s \) decomposable tensors in \( V \otimes \wedge^{k+1} W \), but it cannot be expressed as a linear combination of \( s-1 \) or less decomposable tensors in \( V \otimes \wedge^{k+1} W \).

2.2.2. Contraction maps. The bilinear map \( \wedge^{k+1} W^* \times \wedge^{k+1} W \to k \) given by
\[
\langle f_1 \wedge \cdots \wedge f_k, x_1 \wedge \cdots \wedge x_k \rangle = \det(f_j(x_i))_{0 \leq i, j \leq n},
\]
for \( f_0, \ldots, f_k \in W^* \) and \( x_0, \ldots, x_k \in W \) defines the duality \( \wedge^{k+1} W^* \otimes \wedge^{k+1} W \to k \). This can be naturally extended to the so-called contraction map as follows: Let \( k \) and \( \ell \) be non-negative integers with \( \ell \leq k < \dim V \). Then the contraction map
\[
\bigwedge^{\ell+1} W^* \otimes \bigwedge^{k+1} W \to \bigwedge^{k-\ell} W, \quad f \otimes x \mapsto f \lrcorner x,
\]
is defined as the adjoint of right exterior multiplication, i.e.,
\[
\langle g, f \lrcorner x \rangle = \langle g \wedge f, x \rangle
\]
for all \( g \in \bigwedge^{k-\ell} W^* \).

2.3. Fundamental families of projective varieties. We will start this subsection with three fundamental families of projective varieties we will be dealing with constantly in this paper; namely, Segre varieties, Veronese varieties, and Grassmann varieties. Then we will give the definition of Seger-Grassmann variety and see that the Seger-Grassmann variety parameterizes decomposable partially symmetric and partially skew-symmetric tensors.

Let \( V \) and \( W \) be vector spaces over \( k \) with \( \dim V = m + 1 \) and \( \dim W = n + 1 \). The Segre map \( \text{Seg} : \mathbb{P}V \times \mathbb{P}W \to \mathbb{P}(V \otimes W) \) is the map given by \( \text{Seg}([v], [w]) = [v \otimes w] \) for all \( ([v], [w]) \in \mathbb{P}V \times \mathbb{P}W \). The image of \( \text{Seg} \) is a subvariety of \( \mathbb{P}(V \otimes W) \) of dimension \( m + n \) called the Segre variety and denoted by \( \text{Seg}(\mathbb{P}V \times \mathbb{P}W) \). If \( X \subseteq \mathbb{P}W \) is a projective variety, then we write \( \text{Seg}(\mathbb{P}V \times X) \) for the restriction of \( \text{Seg} \) to \( \mathbb{P}V \times X \).

Let \( d \) be a positive integer, let \( V \) be an \( (m+1) \) vector space over \( k \), and let \( \nu_d : \mathbb{P}V \to \mathbb{P}\text{Sym}_d V \) be the map given by sending \( [v] \) to \( [v^d] \). This map is called the \( d \)th Veronese map. The algebraic variety obtained as the image \( \nu_d(\mathbb{P}V) \) of the Veronese map is called the Veronese variety. Since \( \nu_d(\mathbb{P}V) \) is isomorphic to \( \mathbb{P}V \), it has dimension \( m \). If \( \dim V = 2 \), then \( \nu_d(\mathbb{P}V) \) is called the rational normal curve of degree \( d \).

Let \( V \) be an \( (n+1) \)-dimensional vector space over \( k \) and let \( G(k, \mathbb{P}V) \) denote the Grassmannian of \( k \)-dimensional linear subspaces of \( \mathbb{P}V \). We sometime write \( G(k, n) \) for \( G(k, \mathbb{P}V) \). The Plücker map \( p : G(k, \mathbb{P}V) \to \mathbb{P} \wedge^{k+1} V \) is the map that sends a \( k \) dimensional linear subspace spanned by \( k + 1 \) linearly independent points \( \{v_0, \ldots, v_k\} \) of \( \mathbb{P}V \) to \( [v_0 \wedge \cdots \wedge v_k] \in \mathbb{P} \wedge^{k+1} V \), which describes \( G(k, \mathbb{P}V) \) as a subset of \( \mathbb{P} \wedge^{k+1} V \). It is indeed a subvariety of \( \mathbb{P} \wedge^{k+1} V \) and has dimension \( (k+1)(n-k) \) (see, for more details). Throughout this paper, we will assume that \( G(k, \mathbb{P}V) \) is embedded in \( \mathbb{P} \wedge^{k+1} V \) through the Plücker map.

Let \( V \) and \( W \) be vector spaces over \( k \) with \( \dim V = m + 1 \) and \( \dim W = n + 1 \). Then \( \text{Seg}(\mathbb{P}V \times G(k, \mathbb{P}W)) \) is a subvariety of \( \mathbb{P}(V \otimes \wedge^{k+1} W) \) called the Seger-Grassmann variety. This variety is isomorphic to \( \mathbb{P}V \times G(k, \mathbb{P}W) \), and thus it has dimension \( m + (k+1)(n-k) \).

Note that \( p \in \mathbb{P}(V \otimes \wedge^{k+1} W) \) lies in \( \text{Seg}(\mathbb{P}V \times G(k, \mathbb{P}W)) \) if and only if there exist \( v \in V \) and \( w_0, \ldots, w_k \in W \) such that \( p = [v \otimes w_0 \wedge \cdots \wedge w_k] \). Thus \( \text{Seg}(\mathbb{P}V \times G(k, \mathbb{P}W)) \) can be viewed as the parameter space of decomposable tensors in \( V \otimes \wedge^{k+1} W \).
2.4. Grassmann secant varieties. Let \( W \) be an \((n + 1)\)-dimensional vector space over \( k \), let \( X \subseteq \mathbb{P}W \) be a non-degenerate projective variety of dimension \( d \), and let \( s \in \mathbb{N} \). We write \( X^s \) for the \( s \) copies of \( X \). Define a rational map \( \varphi : X^s \to G(s - 1, \mathbb{P}W) \) by sending an \( s \)-tuple of linearly independent points to the linear subspace spanned by these points.

Let \( m \in \{0, \ldots, s - 1\} \). The \((m, s)\)-Grassmann secant variety of \( X \), denoted \( GS_{m,s}(X) \), is the Zariski closure in \( G(m, \mathbb{P}W) \) of the set of \( m \)-dimensional linear subspaces of \( \mathbb{P}V \), each of which lies in the linear span of \( s \) linearly independent points of \( X \).

The \((m, s)\)-Grassmann secant variety \( GS_{m,s}(X) \) of \( X \) is irreducible and its dimension is bounded by \( \min\{sd + (m + 1)(s - m - 1), (m + 1)(n - m)\} \) from above. The following argument for this is due to Chiantini and Cool in [15]: Let us consider the incidence variety

\[
I = \{(L, P) \in G(m, \mathbb{P}W) \times G(s - 1, \mathbb{P}W) : L \subseteq P \}.
\]

For each \( i \in \{1, 2\} \), let \( \pi_i \) be the projection of \( G(m, \mathbb{P}W) \times G(s - 1, \mathbb{P}W) \) restricted to \( I \) onto the \( i \)-th factor. Then, by definition, \( GS_{m,s}(X) = \pi_1(\pi_2^{-1}(\text{im} \varphi)) \).

Note that if \( d + (s - 1) - n \geq 0 \), then every \((s - 1)\)-dimensional linear subspace of \( \mathbb{P}W \) intersects \( X \) in a non-empty algebraic subset. Thus \( \varphi \) is a dominant rational map. In particular, \( GS_{m,s}(X) \) coincides with \( G(s - 1, \mathbb{P}W) \).

Next assume that \( d + s - n \leq 0 \). Then it follows from the trisecant lemma that \( \pi_2 \) is generically one-to-one (see [19] for more details about the trisecant lemma). The fiber of \( \pi_2 \) over the generic point \( P \) of \( \text{im} \varphi \) is the Grassmann variety of \( m \)-dimensional linear subspaces in \( P \). Therefore, \( \pi_2^{-1}(\text{im} \varphi) \) is irreducible of dimension \( sd + (m + 1)(s - m - 1) \), which implies that \( \dim GS_{m,s}(X) \leq \min\{sd + (m + 1)(s - m - 1), (m + 1)(n - m)\} \).

We say that \( GS_{m,s}(X) \) has the expected dimension if equality holds. Otherwise, \( GS_{m,s}(X) \) is said to be defective. If \( GS_{m,s}(X) \) is defective, then we say that \( X \) is \((m, s)\)-defective. We call the positive integer \( \min\{sd + (m + 1)(s - m - 1), (m + 1)(n - m)\} - \dim GS_{m,s}(X) \) the \((m, s)\)-defect of \( X \) and denote it by \( \delta_{m,s}(X) \).

Remark 2.1. If \( m = 0 \), then \( GS_{0,s}(X) \) is also known as the \( s^{th} \) secant variety of \( X \) and denoted by \( \sigma_s(X) \). We say that \( X \) is \( s \)-defective if \( X \) is \((0, s)\)-defective. If \( X \) is \( s \)-defective, we call the positive integer \( \delta_s(X) = \min\{sd + 1 - 1, n\} - \dim \sigma_s(X) \) the \((s)\)-defect of \( X \).

The following theorem of C. Dionisi and C. Fontanari, which was originally by proved Terracini for Veronese surfaces, describes the connection between the concept of \( X \) being \((m, s)\)-defective and the concept of \( \mathbb{P}^n \times \mathbb{P}^n \) being \( s \)-defective:

**Theorem 2.2** (Terracini [21], Dionisi-Fontanari [14]). Let \( V \) and an \((m + 1)\)-dimensional vector space over \( k \), let \( s \) be a positive integer with \( s \leq n \), and let \( m \in \{0, \ldots, s - 1\} \). Suppose that \( X \subseteq \mathbb{P}W \) is a projective variety. Then the following conditions are equivalent:

(i) \( X \) is \((m, s)\)-defective with \((m, s)\)-defect \( \delta_{m,s}(X) = \delta \).

(ii) \( \text{Seg}(\mathbb{P}V \times X) \) is \( s \)-defective with \( s \)-defect \( \delta_s(\text{Seg}(\mathbb{P}V \times X)) = \delta \).

This equivalence allows one to use tools for studying secant varieties to study Grassmann secant varieties. In the remaining section, we will review some of these tools.

One way of finding the dimension of \( \sigma_s(X) \) is to compute the dimension of the projectivized tangent space \( T_z(\sigma_s(X)) \) to \( \sigma_s(X) \) at a generic point \( z \). The so-called Terracini lemma allows one to describe \( T_z(\sigma_s(X)) \) in terms of a collection of tangent spaces to \( X \) at \( s \) generic points.
Theorem 2.3 (Terracini’s Lemma [22]). Let $V$ be an $(m+1)$-dimensional vector space over $\mathbb{k}$ and let $X$ be a projective variety in $\mathbb{P}V$, let $x_1, \ldots, x_s$ be generic points of $X$, and let $z$ be a generic point of $\langle p_1, \ldots, p_s \rangle$. Then $T_z \sigma_s(X) = \{T_{p_1}X, \ldots, T_{p_s}X\}$.

Remark 2.4. Let $X$, $p_1, \ldots, p_s$, and $z$ be as given in Theorem 2.3. Suppose that $X$ has dimension $d$. Let $I_X \subset \mathbb{k}[x_0, \ldots, x_m]$ be the homogeneous ideal of $X$. Assume that $I_X$ is generated by $F_0, \ldots, F_t \in \mathbb{k}[x_0, \ldots, x_m]$. For each $p_i$, the nullspace of the Jacobian matrix $\text{Jac}(X) = \left( \left. \frac{\partial F_j}{\partial x_i} \right|_{p_i} \right)_{0 \leq i \leq j \leq sm}$ of the generators of $I_X$ evaluated at $p_i$ is $\hat{T}_{p_i}(X)$. In particular, $\hat{T}_{p_i}(X)$ can be represented as an $(m+1) \times (d+1)$ matrix $T_i$ with entries from $\mathbb{k}$. Thus, by Terracini’s lemma, $\hat{T}_i(\sigma_s(X))$ can be obtained as the augmented matrix $T = (A_1 | A_2 | \cdots | A_s)$, and hence the dimension of $\hat{T}_i(\sigma_s(X))$, and hence $\sigma_s(X)$, can be found by computing the rank of $T$. Terracini’s lemma therefore describes the tangent space to a secant variety of $X$ in terms of linear algebra, and thus it makes a computer algebra system feasible to effectively compute the dimension of the secant variety of a quite large variety. In particular, Terracini’s lemma can be used to verify the non-defectivity of the secant varieties.

It can also be used to identify a “potential” defective secant variety and the result of a computation provides strong evidence and gives a dimensional lower bound. It cannot, however, be used as a rigorous proof of its deficiency. The proposition below provides a geometric argument to show that an experimentally suggested defective secant variety is actually defective.

Proposition 2.5. Let $X \subseteq \mathbb{P}^n$ be a non-singular projective variety and let $x_1, \ldots, x_s$ be generic points of $X$. Suppose that there exists a non-singular subvariety $C$ of $X$ passing through $p_1, \ldots, p_s$. Then

\[
\dim(\sigma_s(X)) \leq s(\dim X - \dim C) + \dim \sigma_s(C).
\]

In particular, if

\[
s(\dim X - \dim C) + \dim \sigma_s(C) < \min\{s(\dim(X) + 1), n\} - 1,
\]

then $\sigma_s(X)$ is defective.

Proof. For each $i \in \{1, \ldots, s\}$, choose a subspace $L_i$ of $\hat{T}_{p_i}(X)$ such that $\hat{T}_{p_i}(X) = \hat{T}_{p_i}(C) \oplus L_i$ ($L_i$ is isomorphic to the normal space $N_{p_i}(X/C)$ to $X$ at $p_i$ and has dimension $\dim X - \dim C$). Then since

\[
\sum_{i=1}^s \hat{T}_{p_i}(X) = \sum_{i=1}^s L_i + \sum_{i=1}^s \hat{T}_{p_i}(C),
\]

Terracini’s lemma yields

\[
\dim \sigma_s(X) = \dim \sum_{i=1}^s \hat{T}_{p_i}(X) - 1 \\
\leq \dim \sum_{i=1}^s L_i + \dim \sum_{i=1}^s \hat{T}_{p_i}(C) - 1 \\
\leq s(\dim X - \dim C) + \dim \sigma_s(C).
\]

In particular, if (2.1) holds, then $\dim \sigma_s(X) < \min\{s(\dim(X) + 1), n\} - 1$. So $\sigma_s(X)$ does not have the expected dimension. \hfill \Box

Remark 2.6. Proposition 2.5 is well known among the experts and was used as a standard tool for showing the defectivity of a secant variety in several papers (see, for example, \cite{[S4, S5, S2]}). We however included the proof for the reader’s convenience.
Remark 2.7. Let \( X \subseteq \mathbb{P}^m \) be a projective variety and let \( x_1, \ldots, x_r \) be generic points of \( X \). Inequality (2.1) indicates that, in order to prove the defectivity of \( \sigma_s(X) \) using Proposition 2.5, one needs to find a subvariety \( C \) of \( X \) passing through \( x_1, \ldots, x_r \) whose 4th secant variety has small dimension. For example, if \( X \) is the Veronese surface in \( \mathbb{P}^5 \), then there is a unique conic \( C \) in \( X \) passing through two generic points of \( X \). So, by Proposition 2.5 we obtain

\[
\dim \sigma_s(X) \leq 2(\dim X - \dim C) + \dim \sigma_2(C) = 4 < 5 = \min\{2(2 + 1), 6\} - 1.
\]

Thus \( X \) is 2-defective.

The proposition below is probably well known to experts. We however include a proof for the sake of completeness.

**Proposition 2.8.** Let \( V \) be an \((m + 1)\)-dimensional vector space over \( \mathbb{K} \), let \( W \) be an \((n + 1)\)-dimensional vector space over \( \mathbb{K} \), and let \([v \otimes w_0 \wedge \cdots \wedge w_k] \in \text{Seg}(\mathbb{P}^m \times \mathbb{G}(k, \mathbb{P}^n))\). Then

\[
\dim_{\mathbb{K}} \overline{T}_{[v \otimes w_0 \wedge \cdots \wedge w_k]} \text{Seg}(\mathbb{P}^m \times \mathbb{G}(k, \mathbb{P}^n)) \leq \dim \overline{T}_{[v]} \text{Seg}(\mathbb{P}^m \times \mathbb{G}(k, \mathbb{P}^n)) + \dim \overline{T}_{[w_0]} \text{Seg}(\mathbb{P}^m \times \mathbb{G}(k, \mathbb{P}^n)) + \cdots + \dim \overline{T}_{[w_k]} \text{Seg}(\mathbb{P}^m \times \mathbb{G}(k, \mathbb{P}^n)).
\]

**Proof.** Let \( v' \in V \setminus \{0\} \) be arbitrary. For each \( i \in \{0, \ldots, k\} \), let \( w_i' \in W \setminus \{0\} \) be arbitrary. Consider the following parametric curve

\[
(v + tv') \otimes (w_0 + tw_0') \wedge \cdots \wedge (w_k + tw_k').
\]

The standard calculation of derivative shows that

\[
v' \otimes w_0 \wedge \cdots \wedge w_k + \sum_{i=0}^k v \otimes w_0 \wedge \cdots \wedge w_{i-1} \wedge w_i' \wedge w_{i+1} \wedge \cdots \wedge w_k
\]

is a tangent vector to this parametric curve at \( t = 0 \) unless \( v = v' \) and \( w_i = w_i' \) for each \( i \in \{0, \ldots, k\} \). We thus obtain the desired equality. \( \square \)

**Remark 2.9.** Let \( m, n, \) and \( k \) be non-negative integers. Proposition 2.8 together with Terracini’s lemma enables one to explicitly compute the tangent space to \( \sigma_s(\text{Seg}(\mathbb{P}^m \times \mathbb{G}(k, n))) \) at a given point of \( \text{Seg}(\mathbb{P}^m \times \mathbb{G}(k, n)) \). We will explain how to make this computation below.

Let \( R = \mathbb{K}[e_0, \ldots, e_m] = \bigoplus_{d=0}^m \text{Sym}_d V \) be the polynomial ring with variables \( e_i \) of degree 1 and let \( W = \bigoplus_{i=1}^{m+1} W \) be the exterior algebra with variables \( f_i \) of degree 1. Choose randomly \( s \) points \( p_i = [v_i \otimes w_i, \ldots, w_i, \ldots, \otimes w_{i,k}] \) of \( \text{Seg}(\mathbb{P}^m \times \mathbb{G}(k, n)) \), \( i \in \{0, \ldots, s-1\} \). Next compute, for each \( i \in \{0, \ldots, s-1\} \), the ideal \( I_i \) of \( R \otimes \bigwedge W \) generated by

\[
V \otimes w_{i,0} \wedge \cdots \wedge w_{i,k} + \sum_{j=0}^k v \otimes w_{i,j} \wedge \cdots \wedge w_{i,k+1} \wedge w_{i+1} \wedge \cdots \wedge w_{i,k}.
\]

Then minimize generators of \( I = \sum_{i=0}^{s-1} I_i \). If the resulting minimal generating set for \( I \) consists of \( \ell \)

\[
\min \left\{ s[m + (k + 1)(n - k) + 1], (m + 1) \left( \begin{array}{c} n+1 \\ k+1 \end{array} \right) \right\}
\]

polynomials of bi-degree \( (1, k + 1) \) in \( R \otimes \bigwedge W \), then this means that the dimension of the linear subspace spanned by the tangent spaces to \( \text{Seg}(\mathbb{P}^m \times \mathbb{G}(k, n)) \) at \( p_0, \ldots, p_{s-1} \) equals \( \ell \). By semicontinuity, this implies that if \( q_0, \ldots, q_{s-1} \in \text{Seg}(\mathbb{P}^m \times \mathbb{G}(k, n)) \) are generic, then the dimension of the linear subspace spanned by the \( T_{q_0}, \text{Seg}(\mathbb{P}^m \times \mathbb{G}(k, n)) \)'s is at least \( \ell \). Thus, by Terracini’s lemma, \( \sigma_s(\text{Seg}(\mathbb{P}^m \times \mathbb{G}(k, n))) \) has dimension at least \( \ell \). In
Thus

Suppose that

We proved that (i) implies (ii).

(3.2)

and

particular, if \( \ell = \min\{s[m + (k + 1)(n - k) + 1], (m + 1)\binom{n+1}{k+1}\} \), then we can conclude that \( \sigma_{r}(\text{Seg}(\mathbb{P}^{m} \times \mathbb{G}(k, n))) \) is not defective.

3. The unbalanced case

Let \( X \subseteq \mathbb{P}^{n} \) be a projective variety. The purpose of this section is to introduce the concept of \( \text{Seg}(\mathbb{P}^{m} \times X) \) being “unbalanced” and to show that \( \text{Seg}(\mathbb{P}^{m} \times X) \) is \( s \)-defective for a certain range of \( s \) if \( \text{Seg}(\mathbb{P}^{m} \times X) \) is unbalanced.

Definition 3.1. A triple \((m, n, d)\) of positive integers is said to be unbalanced if \( m > n - d + 1 \). Let \( X \) be a \( d \)-dimensional projective variety in \( \mathbb{P}^{n} \). We say that \( \text{Seg}(\mathbb{P}^{m} \times X) \) is unbalanced if \((m, n, d)\) is unbalanced.

Remark 3.2. It is immediate to see that \((m, n, d)\) is unbalanced if and only if there exists an \( s \in \mathbb{N} \) such that \( n - d + 1 < s \leq \min[m, n] \).

Lemma 3.3. Let \((m, n, d)\) be a triple of positive integers. Then the following conditions are equivalent:

(i) \((m, n, d)\) is unbalanced.

(ii) There exists an \( s \in \mathbb{N} \) with \( s \leq m \) such that

\[
s[m - (s - 1)] + s(n + 1) < \min\{s(m + d + 1), (m + 1)(n + 1)\}.
\]

Proof. The proof uses the following two equalities:

(3.1)

\[
s(m + d + 1) - \{s[m - (s - 1)] + s(n + 1)\} = s[s - (n - d + 1)]
\]

and

(3.2)

\[
(m + 1)(n + 1) - \{s[m - (s - 1)] + s(n + 1)\} = [s - (m + 1)][s - (n + 1)].
\]

First assume that \((m, n, d)\) is unbalanced. Recall that if \( m > n - d + 1 \), then there exists an \( s \in \mathbb{N} \) such that \( n - d + 1 < s \leq \min[m, n] \) (see Remark 3.2). If \( s(m + d + 1) \leq (m + 1)(n + 1) \), then the right hand side of \((3.1)\) is positive because \( s > n - d + 1 \), while if \( s(m + d + 1) > (m + 1)(n + 1) \), then the right hand side of \((3.2)\) is positive because \( s \leq \min[m, n] \leq m \). So we proved that (i) implies (ii).

Conversely, assume the existence of a positive integer \( s \) such that

\[
s[m - (s - 1)] + s(n + 1) < \min\{s(m + d + 1), (m + 1)(n + 1)\}.
\]

Suppose that \( s(m + d + 1) \leq (m + 1)(n + 1) \). Then it follows from \((3.1)\) that \( s[s - (n - d + 1)] > 0 \), which implies that \( s > n - d + 1 \). As \( s(m + d + 1) \leq (m + 1)(n + 1) \), we must have

\[
s \leq (m + 1)(n + 1)/(m + d + 1) < n + 1.
\]

Thus \( n - d + 1 < (m + 1)(n + 1)/(m + d + 1) \), which yields \( m - n + d > 0 \). So

\[
(m + 1) - (m + 1)(n + 1)/(m + d + 1) = (m + 1)(m - n + d)/(m + d + 1) > 0,
\]

and hence we obtain \( s < m + 1 \). Therefore, \( n - d + 1 < s \leq \min[m, n] \).

Next assume that \( s(m + d + 1) > (m + 1)(n + 1) \). Then it follows from \((3.2)\) that \( [s - (m + 1)][s - (n + 1)] > 0 \), which implies that \( s \leq \min[m, n] \) or \( s > \max[m + 1, n + 1] \). Since \( s \leq m \) by assumption, we must have \( s \leq \min[m, n] \). So it remains only to show that \( n - d + 1 < s \) under this assumption.

Since \( s(m + d + 1) > (m + 1)(n + 1) \), we obtain

\[
(m + 1)(n + 1)/(m + d + 1) \leq s \leq \min[m, n] < m + 1.
\]
It is immediate to show \( n - m < d \). So, in order to prove that \( n - d + 1 < m \), it suffices to show that \( m \neq n - d + 1 \). Assume for the contradiction that \( m = n - d + 1 \). Then we have

\[
(m + 1)(n + 1)/(m + d + 1) \leq \min[m, n] = n - d + 1,
\]

from which one can immediately derive \( d(d + m - n) \leq 0 \). Thus we have \( d \leq n - m \). This contradicts, however, \( n - m < d \), and so we proved that \( n - d + 1 < s \leq \min[m, n] \). Thus \( (m, n, d) \) is unbalanced by Remark \[3.2\].

**Theorem 3.4.** Let \((m, n, d)\) be unbalanced. Then \( \sigma_s(\text{Seg}(P^m \times X)) \) is defective if and only if \( n - d + 1 < s \leq \min[m, n] \).

**Proof.** Suppose that \( n - d + 1 < s \leq \min[m, n] \). Let \( y_1, \ldots, y_s \in \text{Seg}(P^m \times X) \) be generic and let \( x_1 \in P^m \times X \) such that \( \text{Seg}(x_i) = y_i \) for each \( i \in \{1, \ldots, s\} \). Denote by \( \pi \) the projection from \( P^m \times X \) onto \( P^m \). Then, by the choice of \( x_1, \ldots, x_s \), the \( \pi(x_i) \)'s span an \((s - 1)\)-plane \( P \) in \( P^m \). This means that \( P \times X \) contains the \( x_i \)'s, and hence \( \text{Seg}(P \times X) \) passes through \( y_1, \ldots, y_s \). Therefore, by Proposition \[2.5\], we obtain

\[
\dim \sigma_s(\text{Seg}(P^m \times X)) \leq s[m - (s - 1)] + \dim \sigma_s(\text{Seg}(P \times X))
\]

\[
\leq s[m - (s - 1)] + \min[s(s - 1) + d], s(n + 1) - 1
\]

\[
\leq s[m - (s - 1)] + s(n + 1) - 1.
\]

Since \( n - d + 1 < s \leq \min[m, n] \), Lemma \[3.3\] yields

\[
s[m - (s - 1)] + s(n + 1) - 1 < \min[s(m + d + 1), (m + 1)(n + 1)] - 1,
\]

from which, the defectivity of \( \sigma_s(\text{Seg}(P^m \times X)) \) follows.

To prove the other direction, we show that if either (a) \( s > \min[m, n] \) or (b) \( s \leq n - d + 1 \), then \( \sigma_s(\text{Seg}(P^m \times X)) \) has the expected dimension.

(a) First we prove that if \( s > \min[m, n] \), then \( s \geq (m + 1)(n + 1)/(m + d + 1) \). If \( m \leq n \), then \( s > m \) by assumption. Thus

\[
s(m + d + 1) \geq (m + 1)(m + d + 1) \geq (m + 1)(n + 1),
\]

because \( m > n - d + 1 \). Therefore, \( s \geq (m + 1)(n + 1)/(m + d + 1) \).

Now assume \( m > n \). Then \( s > \min[m, n] = n \), and so \( s(m + d + 1) \geq (n + 1)(m + d + 1) > (n + 1)(m + 1) \). Therefore, \( s \geq (m + 1)(n + 1)/(m + d + 1) \).

Since \((m, n, d)\) is unbalanced, \( \min[m, n] \geq n - d + 1 \). Thus if \( s > \min[m, n] \), then \( \text{codim}(X) + 1 = n - d + 1 < \min[m, n] < s \). From Lemmas 2.2 and 2.3 in \[12\], it follows therefore that \( \sigma_s(\text{Seg}(P^m \times X)) = \sigma_s(\text{Seg}(P^m \times P^n)) = P^{(m+1)(n+1)−1} \). In particular, \( \sigma_s(\text{Seg}(P^m \times X)) \) has the expected dimension.

(b) Assume that \( s \leq n - d + 1 \). Then

\[
(m + 1)(n + 1) - s(m + d + 1) \geq (m + 1)(n + 1) - (n - d + 1)(m + d + 1)
\]

\[
= d(m - n + d)
\]

\[
> 0,
\]

because \((m, n, d)\) is unbalanced. So \( s < (m + 1)(n + 1)/(m + d + 1) \), and thus it is enough to show that if \( s = n - d + 1 \), then \( \sigma_s(\text{Seg}(P^m \times X)) \) has the expected dimension.
Remark \[ \sigma \] determined whether or not result obtained by Ballico, Bernardi, Catalisano, and Chinitini in [9]. More precisely, they have shown that every intersection of \( \sigma_s(\text{Seg}(\mathbb{P}^m \times \mathbb{P}^n)) \) with \( d_1 = 1 \) in [12]. Theorem 3.4 was probably known to the experts, but we were not aware of any proof in the literature. We thus decided to include the proof.

It is worth noticing that there is a significant overlap between Theorem 3.4 and the result obtained by Ballico, Bernardi, Catalisano, and Chinitini in [9]. More precisely, they determined whether or not \( \sigma_s(\text{Seg}(\mathbb{P}^m \times \mathbb{P}^n)) \) (not necessarily unbalanced) is defective for each \( (m, n, s) \) except for the \( (m, n, s) \)'s with \( m < s < n \). However, their result did not complete the classification of defective secant varieties to unbalanced Segre-Grassmann varieties.

**Remark 3.5.** Let \( X \subseteq \mathbb{P}^m \) be a \( d \)-dimensional projective variety and let \( s \) be an integer with \( n - d + 2 \leq s \leq \min(m, n) \). Suppose that \( (m, n, d) \) is unbalanced. Then, by definition, we have \( \dim X + 1 = n - d + 1 < s \). Thus, by Lemmas 2.2 and 2.3 in [12], \( \sigma_s(\text{Seg}(\mathbb{P}^m \times \mathbb{P}^n)) = \sigma_s(\text{Seg}(\mathbb{P}^m \times \mathbb{P}^n)) \) in particular, the codimension of \( \sigma_s(\text{Seg}(\mathbb{P}^m \times \mathbb{P}^n)) \) is \( (m + 1 - s)(n + 1 - s) \) instead of \( \max\{0, (m + 1)(n + 1) - s(m + d + 1)\} \). This is another proof of the defectivity of \( \sigma_s(\text{Seg}(\mathbb{P}^m \times \mathbb{P}^n)) \) for \( n - d + 2 \leq s \leq \min(m, n) \).

4. **Another infinite family of defective Segre-Grassmann varieties**

The purpose of this section is to show that \( \text{Seg}(\mathbb{P}^2 \times \mathbb{G}(1, 4\ell + 2)) \) is \( \ell \geq 0 \), is defective. If \( s = 3\ell + 2 \), then \( \sigma_s(\text{Seg}(\mathbb{P}^2 \times \mathbb{G}(1, 4\ell + 2))) \subseteq \mathbb{P}(\mathbb{G}(1, 4\ell + 2)) \) is expected to coincide with \( \mathbb{P}^{3(\ell + 2)} \). Our goal is to prove that this is actually not the case. This family is inspired by the recent paper [13] by Ottaviani. In the next paragraph, we will review relevant results of his paper.

In the paper [20], Strassen showed that \( \text{Seg}(\mathbb{P}^2 \times \mathbb{P}^{n-1} \times \mathbb{P}^{n-1}) \) is \( (3n - 2)/2 \)-defective. More precisely, if \( s = (3n - 2)/2 \), then \( \sigma_s(\text{Seg}(\mathbb{P}^2 \times \mathbb{P}^{n-1} \times \mathbb{P}^{n-1})) \subseteq \mathbb{P}^{3(\ell + n - 2)/2} \). Strassen proved, however, that it does not by showing that \( \sigma_s(\text{Seg}(\mathbb{P}^2 \times \mathbb{P}^{n-1} \times \mathbb{P}^{n-1})) \) is a hypersurface. He also gave an equation for \( \sigma_s(\text{Seg}(\mathbb{P}^2 \times \mathbb{P}^{n-1} \times \mathbb{P}^{n-1})) \), which is now called the Strassen equation. In [13], Ottaviani reconstructed the Strassen equation using a linear algebra approach. In the same paragraph, he used the same approach (with a slight modification) to show that \( \text{Seg}(\mathbb{P}^2 \times \mathbb{P}^{n-1} \times \mathbb{P}^{n-1}) \) with \( n \) even is \( (3n/2 - 1) \)-defective. Note that \( \sigma_s(\text{Seg}(\mathbb{P}^2 \times \mathbb{P}^{n-1} \times \mathbb{P}^{n-1})) \subseteq \mathbb{P}^{3(\ell + n - 2)/2} \) is expected to fill \( \mathbb{P}^{3(\ell + n - 2)/2} \) if \( s = 3\ell + 2 \). But he showed that \( \sigma_s(\text{Seg}(\mathbb{P}^2 \times \mathbb{P}^{n-1} \times \mathbb{P}^{n-1})) \) is a hypersurface and discovered an equation for \( \text{Seg}(\mathbb{P}^2 \times \mathbb{P}^{n-1} \times \mathbb{P}^{n-1}) \). In this section, we extend Ottaviani’s approach to show the defectivity of \( \sigma_s(\text{Seg}(\mathbb{P}^2 \times \mathbb{G}(1, 4\ell + 2))) \) if \( s = 3\ell + 2 \).

Let \( V \) be a three-dimensional vector space over \( \mathbb{K} \) with basis \( \{e_0, \ldots, e_2\} \) and let \( W_\ell \) be a \((4\ell + 3)\)-dimensional vector space over \( \mathbb{K} \) with basis \( \{f_0, \ldots, f_{4\ell + 2}\} \). For a fixed \( \sigma_{w_1} \wedge w_2 \in V \otimes (V \wedge)^2 \), define a bilinear map \( \langle \cdot, \cdot \rangle : V \otimes W_\ell^* \rightarrow (\wedge^2 V \otimes W_\ell^*) \) by \( \langle v, w^* \rangle = v \wedge w^* \wedge (w_1 \wedge w_2) \). We denote by \( A_{w_1 \wedge w_2}(v \otimes w^*) \) the linear transformation from \( V \otimes W_\ell^* \) to \( \wedge^2 V \otimes W_\ell^* \). We define \( A_{w_1 \wedge w_2}(v \otimes w^*) \) as \( \langle v, w^* \rangle \).
Since $V \otimes \bigwedge^2 W_\ell$ is generated by tensors of the form $u \otimes w_1 \wedge w_2$, the linear transformation $A_{\phi, w_1, w_2} : V \otimes W_\ell^* \rightarrow \bigwedge^2 V \otimes W_\ell$ can be canonically extended to the general member $\phi$ of $V \otimes \bigwedge^2 W_\ell$. We denote by $A_\phi$ such an extended linear transformation.

Note that if $\phi = u \otimes w_1 \wedge w_2 \in V \otimes \bigwedge^2 W$, then it is straightforward to show that $\text{im}(A_\phi) = u \wedge V \otimes w_1 + u \wedge V \otimes w_2$.

Thus if $\phi \in V \otimes \bigwedge^2 W_\ell$ has rank one, then $\text{rank}(A_\phi) = 4$. In particular, if the rank of $\phi \in V \otimes \bigwedge^2 W_\ell$ is $s$, then $\text{rank}(A_\phi) \leq 4s$. Moreover, if $[\phi] \in \sigma_j(\mathbb{P}(V \times \mathbb{G}(1, 4\ell + 2))$, then, by semi-continuity, we have $\text{rank}(A_\phi) \leq 4s$.

**Lemma 4.1.** Let $\phi_\ell \in V \otimes \bigwedge^2 W_\ell$ be generic of rank $3\ell + 3$. Then $A_{\phi_\ell}$ has rank $3(4\ell + 3)$.

**Proof.** It is sufficient to show the existence of a rank $3\ell + 3$ tensor $\phi_\ell \in V \otimes \bigwedge^2 W_\ell$ such that $\text{rank}(A_{\phi_\ell}) = 3(4\ell + 3)$. The proof is by induction on $\ell$.

Let $\ell = 0$ and let $\phi_0 = e_0 \otimes f_0 \wedge f_1 + e_1 \otimes f_1 \wedge f_2 + e_2 \otimes f_2 \wedge f_0$. A straightforward calculation shows that $\text{rank}(A_{\phi_0}) = 9$.

Now assume that, for some $\ell \geq 0$, there exists a rank $3\ell + 3$ tensor $\phi_\ell \in V \otimes \bigwedge^2 W_\ell$ such that $\text{rank}(A_{\phi_\ell}) = 3(4\ell + 3)$. Let $U = \text{Span}(f_{4\ell+3}, \ldots, f_{4\ell+6})$. Then

$$V \otimes \bigwedge^2 W_{\ell+1} = \left( V \otimes \bigwedge^2 W_\ell \right) \oplus \left( V \otimes \bigwedge^2 U \right).$$

So, if we can find a rank three tensor $\psi \in V \otimes \bigwedge^2 U$ such that $A_{\phi_\ell} : V \otimes U^* \rightarrow \bigwedge^2 V \otimes U$ has rank 12, then $\phi_{\ell+1} = \phi_\ell + \psi \in V \otimes \bigwedge^2 W_{\ell+1}$ is the desired rank $3(\ell + 1) + 3$ tensor, because then we have $A_{\phi_{\ell+1}} = A_{\phi_\ell} \oplus A_{\psi}$ and $\text{rank}(A_{\phi_{\ell+1}}) = \text{rank}(A_{\phi_\ell}) + \text{rank}(A_{\psi}) = 3(4\ell + 3) + 12 = 3(4\ell + 4) + 1$. For instance, if

$$\psi = e_0 \otimes (f_{4\ell+3} \wedge f_{4\ell+4}) + e_1 \otimes (f_{4\ell+5} \wedge f_{4\ell+6}) + e_2 \otimes (f_{4\ell+3} + f_{4\ell+6}) \wedge (f_{4\ell+4} + f_{4\ell+5}),$$

then some tedious manipulation yields $\text{rank}(A_{\phi_\ell}) = 12$, as required.

**Theorem 4.2.** Let $s = 3\ell + 2$. Then $\sigma_j(\mathbb{P}(\mathbb{P}^2 \times \mathbb{G}(1, 4\ell + 2)))$ is defective.

**Proof.** If $s = 3\ell + 2$, then it is immediate to show that

$$s[2 + 2(4\ell + 2 - 1)] - 1 > 3\binom{4\ell + 3}{2} - 1.$$

Therefore, $\dim(\mathbb{P}(\mathbb{P}^2 \times \mathbb{G}(1, 4\ell + 2))) \geq \dim(\mathbb{P}(\bigwedge^2 W_\ell))$. So the expected dimension of $\sigma_j(\mathbb{P}(\mathbb{P}^2 \times \mathbb{G}(1, 4\ell + 2)))$ is $3\binom{4\ell + 3}{2} - 1$. On the other hand, if $\phi \in V \otimes \bigwedge^2 W_\ell$ has rank $s$, then $\text{rank}(A_{\phi}) \leq 4(3\ell + 2) < 3(4\ell + 3)$. Therefore, it follows from Lemma 4.1 that the generic point of $\mathbb{P}(V \otimes \bigwedge^2 W)$ does not lie in $\sigma_j(\mathbb{P}(\mathbb{P}^2 \times \mathbb{G}(1, 4\ell + 2)))$, or $\sigma_j(\mathbb{P}(\mathbb{P}^2 \times \mathbb{G}(1, 4\ell + 2))) \neq \mathbb{P}(V \otimes \bigwedge^2 W_\ell)$. Thus $\sigma_j(\mathbb{P}(\mathbb{P}^2 \times \mathbb{G}(1, 4\ell + 2)))$ does not have the expected dimension. This completes the proof.

**5. Sporadic cases**

This section will be devoted to proving the existence of four isolated defective secant varieties of Segre-Grassmann varieties. More precisely, we will show that $\sigma_j(\mathbb{P}^m \times \mathbb{G}(k, n))$ does not have the expected dimension if $(m, n, k, s) \in \{(2, 5, 1, 4), (2, 7, 1, 5), (1, 5, 2, 3), (2, 5, 2, 5)\)$. 

Lemma 5.1. Let $s, k \in \mathbb{N}$ with $s \geq 2$. Then there exists a $(k + 1)$-ple Veronese embedding of $\mathbb{P}^{s-2}$ in $G(k, (s-1)(k+1)-1)$ passing through generic $s$ points of $G(k, (s-1)(k+1)-1)$.

Proof. Let $W$ be an $(s-1)(k+1)$-dimensional vector space over $\mathbb{k}$ with basis \{ $f_0, \ldots, f_{(s-1)(k+1)-1}$ \} and let $W_0, \ldots, W_{s-1}$ be generic two-dimensional subspaces of $W$. Then a simple parameter count shows that there exists a collineation $\Phi$ of $G(k, (s-1)(k+1)-1)$ such that

$$\Phi(W_i) = \begin{cases} \text{Span}(f_{i(k+1)+1}, \ldots, f_{i(k+1)+k}) & \text{if } i \in \{0, \ldots, s-2\} \\ \text{Span}(\sum_{i=0}^{s-2} f_{i(k+1)+1}, \ldots, \sum_{i=0}^{s-2} f_{i(k+1)+k}) & \text{if } i = s-1. \end{cases}$$

We may assume, therefore, that $W_i = \text{Span}(f_{i(k+1)+1}, \ldots, f_{i(k+1)+k})$ for each $i \in \{0, \ldots, s-3\}$ and $W_{s-2} = \text{Span}(\sum_{i=0}^{s-2} f_{i(k+1)+1}, \ldots, \sum_{i=0}^{s-2} f_{i(k+1)+k})$.

Consider the following family of $(k+1)$-dimensional subspaces of $W$ parameterized by $\mathbb{P}^{s-2}$:

$$\left\{ \text{Span}\left(\sum_{i=0}^{s-2} t_i f_{i(k+1)+1}, \ldots, \sum_{i=0}^{s-2} t_i f_{i(k+1)+k}\right) \left| \begin{array}{l} t_0 : \cdots : t_{s-2} \in \mathbb{P}^{s-2} \end{array} \right. \right\}.$$

Since this family contains $W_0, \ldots, W_{s-1}$, we can use it to see that there is a Veronese embedding of $\mathbb{P}^{s-2}$ with $O(k+1)$ in $G(k, (s-1)(k+1)-1)$ passing through $W_0, \ldots, W_{s-1}$. □

Theorem 5.2. Let $m, s, k \in \mathbb{N}$ with $s \geq 2$ and $m \geq 3$. If the following inequality holds:

$$s((s-1)(k+1)-1-(s-2)) + \dim \sigma_s(\text{Seg}(\mathbb{P}^m \times v_{k+1}(\mathbb{P}^{s-2}))) < \min \left\{ s(m + (k + 1)^2(s-2)+1), (m+1)\left(\frac{(s-1)(k+1)}{k+1}\right) \right\} - 1,$$

then $\sigma_s(\text{Seg}(\mathbb{P}^m \times G(k+1, (s-1)(k+1)-1)))$ is defective. In particular, $\sigma_s(\text{Seg}(\mathbb{P}^m \times G(k+1, (s-1)(k+1)-1)))$ is defective for each $(m, k, s) \in \{(2, 1, 4), (2, 1, 5), (1, 2, 3)\}$.

Proof. Let $y_1, \ldots, y_s$ be generic points of $\text{Seg}(\mathbb{P}^m \times G(k+1, (s-1)(k+1)-1))$ and let $x_i \in \mathbb{P}^m \times G(k+1, (s-1)(k+1)-1)$ with $\text{Seg}(x_i) = y_i$ for each $i \in \{1, \ldots, s\}$. We write $\pi$ for the projection from $\mathbb{P}^m \times G(k+1, (s-1)(k+1)-1)$ onto its second factor.

By Lemma 5.1, there exists a $(k+1)$-ple Veronese embedding $v_{k+1}(\mathbb{P}^{s-2})$ of $\mathbb{P}^{s-2}$ in $G(k+1, (s-1)(k+1)-1)$ that passes through $\pi(x_1), \ldots, \pi(x_s)$. Thus $\mathbb{P}^m \times v_{k+1}(\mathbb{P}^{s-2})$ contains $x_1, \ldots, x_s$, and hence $\text{Seg}(\mathbb{P}^m \times v_{k+1}(\mathbb{P}^{s-2}))$ passes through $y_1, \ldots, y_s$. Therefore, by Proposition 2.3,

$$\dim \sigma_s(\text{Seg}(\mathbb{P}^m \times G(k+1, (s-1)(k+1)-1))) \leq s((s-1)(k+1)-1-(s-2)) + \dim \sigma_s(\text{Seg}(\mathbb{P}^m \times v_{k+1}(\mathbb{P}^{s-2}))).$$

Thus if $(m, k, s)$ satisfies (5.1), then $\sigma_s(\text{Seg}(\mathbb{P}^m \times G(k+1, (s-1)(k+1)-1)))$ is defective. It is known that

$$\dim \sigma_s(\text{Seg}(\mathbb{P}^m \times v_{k+1}(\mathbb{P}^{s-2}))) = \begin{cases} 17 & \text{if } (m, k, s) = (2, 1, 4) \text{ (see [3])} \\ 28 & \text{if } (m, k, s) = (2, 1, 5) \text{ (see [11, 13])} \\ 7 & \text{if } (m, k, s) = (1, 2, 3) \text{ (see [6])}. \end{cases}$$

A straightforward computation shows that $(2, 1, 4), (2, 1, 5),$ and $(1, 2, 3)$ satisfy (5.1). Thus $\sigma_s(\text{Seg}(\mathbb{P}^m \times G(k+1, (s-1)(k+1)-1)))$ is defective if $(m, k, s) \in \{(2, 1, 4), (2, 1, 5), (1, 2, 3)\}$. □

Let $V$ be a three-dimensional vector space over $\mathbb{k}$ and let $W$ be a six-dimensional vector space over $\mathbb{k}$ with basis \{ $f_0, \ldots, f_3$ \}. For a given $v \otimes (w_1 \wedge w_2 \wedge w_3) \in V \otimes \wedge^3 W$, consider the alternating bilinear map $\langle \ , \ : W^* \times W^* \longrightarrow V \otimes W$ given by $\langle u'_1, u'_2 \rangle = v \otimes (u'_1 \wedge u'_2) \wedge (w_1 \wedge w_2 \wedge w_3)$. Let $A_{\otimes \ominus (w_1 \wedge w_2 \wedge w_3)}$ denote the linear transformation from $\wedge^2 W^* \otimes W \otimes W$. 

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such that $A_{\eta}(w_1 \land w_2 \land w_3)(u_1^i \land u_2^j) = \langle u_1^i, u_2^j \rangle$. Since the tensors of the form $v \otimes (w_1 \land w_2 \land w_3)$ generate $V \otimes \wedge^3 W$, one can naturally extend $A_{\eta}(w_1 \land w_2 \land w_3)$ to a linear transformation $A_{\phi} : \wedge^2 W \to V \otimes W$ for every $\phi \in V \otimes \wedge^3 W$.

Note that if $v \otimes (w_1 \land w_2 \land w_3) \in V \otimes \wedge^3 W$ is non-zero, then $\text{im} A_{\eta}(w_1 \land w_2 \land w_3) = \text{Span}(v \otimes w_1, v \otimes w_2, v \otimes w_3)$, and thus rank $A_{\eta}(w_1 \land w_2 \land w_3) = 3$, which means that if rank $\phi = s$, then rank $A_{\phi} \leq 3s$.

**Theorem 5.3.** $\sigma_3(\text{Seg}(\mathbb{P}^2 \times \mathbb{G}(2, 5)))$ is defective.

**Proof.** Since $5(\dim \text{Seg}(\mathbb{P} \times \mathbb{G}(2, 5))) + 1 = \dim V \otimes \wedge^3 W = 60$, the expected dimension of $\sigma_3(\text{Seg}(\mathbb{P} \times \mathbb{G}(2, 5)))$ is 59. So it is sufficient to prove that $\dim \sigma_3(\text{Seg}(\mathbb{P} \times \mathbb{G}(2, 5))) \leq 58$.

Consider $\phi_1 = e_0 \otimes f_0 \land f_1 \land f_2$, $\phi_2 = e_1 \otimes f_0 \land f_3 \land f_4$, $\phi_3 = e_2 \otimes f_1 \land f_3 \land f_5$, $\phi_4 = (e_0 + e_1) \otimes f_2 \land f_3 \land f_5$, and $\phi_5 = (e_1 + e_2) \otimes (f_1 + f_2) \land (f_0 + f_3) \land (f_4 + f_5)$. Let $\phi = \sum_{i=1}^5 \phi_i$. Then an explicit computation shows that $A_{\phi}$ has rank 15. By semi-continuity, if $[\phi] \in \sigma_3(\text{Seg}(\mathbb{P} \times \mathbb{G}(2, 5)))$ is generic, then rank $A_{\phi} = 15$, and thus $\text{im} A_{\phi}$ is a linear subspace of $\mathbb{P}(V \otimes W)$ of dimension 14.

Conversely, if $H$ is a generic linear subspace of $\mathbb{P}(V \otimes W)$ of dimension 14 (this means that the codimension of $H$ in $\mathbb{P}(V \otimes W)$ is three), then, for each $\phi \in V \otimes W$, there exists a unique plane $Q_\phi$ in $\mathbb{P}W$ such that $H$ intersects the fiber $\text{Seg}([\phi] \times \mathbb{P}W) = \mathbb{P}W$ of the projection from $\text{Seg}(\mathbb{P} \times \mathbb{P}W)$ to $\mathbb{P}W$ over $\phi$ in $\text{Seg}([\phi] \times Q_\phi)$. Since $\text{Seg}(\mathbb{P} \times \mathbb{P}W) \cap H \subset H$ is non-degenerate, one can find points $[v_1], \ldots, [v_5] \in \mathbb{P}V$ and planes $Q_1 = \langle [w_{1,1}, [w_{2,1}, [w_{3,1}]]], \ldots, Q_5 = \langle [w_{1,5}, [w_{2,5}, [w_{3,5}]]] \rangle$ in $\mathbb{P}W$ such that the $\text{Seg}([\phi] \times Q_\phi)$'s span $H$, and thus there exists a tensor $\phi \in V \otimes \wedge^3 W$ of rank 5 such that $H = \text{im} A_{\phi}$. As a result, if $\phi = \sum_{i=1}^5 \delta_i \otimes ([w_{1,i} \land w_{2,i} \land w_{3,i}] \in V \otimes \wedge^3 W$ is generic of rank 5, then we may assume that $A_{\phi}$ has rank 15 and that there exists a unique plane $Q_{[v]} = \langle [w_{1,v}], [w_{2,v}], [w_{2,v}] \rangle$ in $\mathbb{P}W$ for each $[v] \in \mathbb{P}V$ such that $\text{im} A_{\phi}$ intersects $\text{Seg}([\phi] \times Q_{[v]})$.

Let $[s_0 : s_1 : s_2]$ and $[t_0 : \cdots : t_5]$ be homogeneous coordinates of $\mathbb{P}V$ and $\mathbb{P}W$ respectively. For a generic rank 5 tensor $\phi \in V \otimes \wedge^3 W$, $\text{im} A_{\phi} \subset \mathbb{P}(V \otimes W)$ is a linear subspace of codimension 3, the intersection of $\text{im} A_{\phi}$ with $\text{Seg}(\mathbb{P} \times \mathbb{P}W)$ may be identified with the algebraic variety in $\mathbb{P} \times \mathbb{P}W$ defined by three generic biforms $\sum_{i,j,k} \gamma_{i,j,k} s_i t_j$ of bidegree $(1, 1)$, where $i \in \{0, 1, 2\}$ and $\gamma_{i,j,k} \in k$. Then the plane in $\mathbb{P}W$ obtained from $\text{im} A_{\phi} \cap \text{Seg}([s_0 : s_1 : s_2] \times \mathbb{P}W)$ by projecting onto $\mathbb{P}W$ is defined by the system of linear equations in $t_0, \ldots, t_5$ whose matrix coefficient is

$$B(s_0, s_1, s_2) = \begin{pmatrix} \sum_{k=0}^2 \gamma_{k,j,s_i} \end{pmatrix}_{0 \leq i \leq 2, 0 \leq j \leq 5}.$$

The matrix $B(s_0, s_1, s_2)$ can be viewed as a generic element of $H^0(\mathbb{P}V, O_{\mathbb{P}V}(1)^{\otimes 3})$, and thus it defines an embedding $\eta : \mathbb{P}V \to \mathbb{G}(2, \mathbb{P}W^*)$. Let $\delta : \mathbb{G}(2, \mathbb{P}W^*) \to \mathbb{G}(2, \mathbb{P}W)$ be the isomorphism obtained from the isomorphism $\wedge^3 W^* \cong \wedge^3 W \to \wedge^3 W$ induced by the bilinear map $\lambda : \wedge^3 W \times \wedge^3 W \to k$ given by $\lambda(f_0 \land \cdots \land f_3) \mapsto \lambda$. Then $(\delta \circ \eta(\mathbb{P}V)$ is a triple Veronese embedding of $\mathbb{P}V$ in $\mathbb{G}(2, \mathbb{P}W) \subset \mathbb{P} \wedge^3 W$ that passes through $q_1 = [w_{1,1} \land w_{2,1} \land w_{3,1}], \ldots, q_5 = [w_{1,5} \land w_{2,5} \land w_{3,5}]$.

Let $C$ be the conic in $\mathbb{P}V$ passing through $[v_1], \ldots, [v_5]$. Then $(\delta \circ \eta(C)$ is a rational normal curve of degree 6 that passes through $q_1, \ldots, q_5$. Let $\iota : C \to \mathbb{P}V$ be the inclusion. Then the image of $C$ under the composite of $\sigma(\iota, \delta \circ \eta) : C \to \text{Seg}(\mathbb{P} \times \mathbb{G}(2, \mathbb{P}W)$ is a rational normal curve of degree $8 = 2 + 2 \cdot 3$ passing through $[v_1 \otimes (w_{1,1} \land w_{2,1} \land$
\( w_{3,1}, \ldots, [v_5 \otimes (w_{1,5} \wedge w_{2,5} \wedge w_{3,5})] \in \mathbb{P}V \times G(2, \mathbb{P}W) \). Thus, by Proposition 2.5,

\[
\dim \sigma_5(\text{Seg}(\mathbb{P}V \times G(2, \mathbb{P}W))) \leq 5[\dim \text{Seg}(\mathbb{P}V \times G(2, \mathbb{P}W)) - \dim C] + \dim \sigma_5(C) = 5[2 + 3(5 - 2) - 1] + 8 = 58,
\]

and so \( \sigma_5(\text{Seg}(\mathbb{P}V \times G(2, \mathbb{P}W))) \) does not have the expected dimension. Therefore the proof is completed.

We performed a considerable amount of experimentation to find further defective secant varieties of Segre-Grassmann varieties. We also attempted to generalize the ideas of Sections 4 and 5. However, no further examples of defective secant varieties of Segre-Grassmann varieties were discovered. We thus would like to conclude this section with the following question:

**Question 5.4.** Are there any quadruples \((m, n, k, s)\), other than the ones in Table 2, such that \( \sigma_s(\mathbb{P}^m \times G(k, n)) \) are defective?

6. **The Secant Variety \( \sigma_{3\ell+2}(\text{Seg}(\mathbb{P}^2 \times G(1, 4\ell + 2))) \) is a Hypersurface**

In Sections 3 and 5, we showed that the secant varieties corresponding to Rows (i)-(vi) of Table 2 are defective. The goal of this section is to prove that these secant varieties have the same codimensions as indicated in Table 3.

As we saw in Remark 3.6 if \((m, n, k)\) is unbalanced and if \(n - d + 2 \leq s \leq \min\{m, \binom{n+1}{k+1} - 1\}\), then \( \sigma_s(\mathbb{P}^m \times G(k, n)) = \sigma_s(\mathbb{P}^m \times \mathbb{P}(\mathbb{P}^{G_W})) \). In particular,

\[
\text{codim} \left( \sigma_s(\mathbb{P}^m \times G(k, n)), \mathbb{P}(\mathbb{P}^{G_W}) \right) = (m + 1 - s) \binom{n + 1}{k + 1}.
\]

We also showed that if \((m, n, k, s) \in \{(2, 5, 1, 4), (2, 7, 1, 5), (1, 5, 2, 3), (2, 5, 2, 5)\}\), then the codimensions of \( \sigma_s(\mathbb{P}^m \times G(k, n)) \) are bounded by 3, 10, 8, and 1 (instead of 1, 9, 7, and 0) from below respectively. One can use the randomized algorithm as described in Remark 2.9 to show that 3, 10, 8, and 1 are actually the codimensions of \( \sigma_s(\mathbb{P}^m \times G(k, n)) \) for these quadruples. It remains therefore only to show that \( \sigma_{3\ell+2}(\text{Seg}(\mathbb{P}^2 \times G(1, 4\ell + 2))) \) are hypersurfaces for all non-negative integers \( \ell \). In Theorem 4.2, we proved that these secant varieties have at least codimension one, and hence it is sufficient to show that they have at most codimension one. To do so, we use induction on \( \ell \).

Let \( V \) be a three-dimensional vector space over \( \mathbb{k} \) with basis \( \{e_0, e_1, e_2\} \). For each non-negative integer \( \ell \) and let \( W \) be a \((4\ell + 3)\)-dimensional vector space over \( \mathbb{k} \) with basis \( \Lambda = \{f_0, \ldots, f_{4\ell+2}\} \). The goal of this section is to show that \( \sigma_{3\ell+2}(\text{Seg}(\mathbb{P}V \times G(1, \mathbb{P}W))) \) has dimension \( 3^{(4\ell+3) \over 2} - 2 \). By Terracini's lemma, to prove that the dimension of \( \sigma_{3\ell+2}(\text{Seg}(\mathbb{P}V \times G(1, \mathbb{P}W))) \) is \( 3^{(4\ell+3) \over 2} - 2 \), it is sufficient to show that \( \sum_{i=0}^{3\ell+1} \text{tr}_{p_{3\ell+1}} \text{Seg}(\mathbb{P}V \times G(1, \mathbb{P}W)) \) has dimension \( 3^{(4\ell+3) \over 2} - 1 \) for generic \( p_0, \ldots, p_{3\ell+1} \in \text{Seg}(\mathbb{P}V \times G(1, \mathbb{P}W)) \). Let \( T = \bigcup_{i=0}^{3\ell+1} \text{tr}_{p_{3\ell+1}} \text{Seg}(\mathbb{P}V \times G(1, \mathbb{P}W)) \). Then, to prove that \( \text{dim} \sum_{i=0}^{3\ell+1} \text{tr}_{p_{3\ell+1}} \text{Seg}(\mathbb{P}V \times G(1, \mathbb{P}W)) = 3^{(4\ell+3) \over 2} - 1 \), it is equivalent to showing that there is a unique hyperplane of \( \mathbb{P}(V \otimes \Lambda^2 W) \) containing \( T \) or \( \text{dim} H^0(\mathbb{P}(V \otimes \Lambda^2 W), \mathcal{O}(1)) = 1 \).

For each \( i \in [0, 1] \), let \( \Lambda_i = \{f_{4\ell+2-i-1}, \ldots, f_{4\ell+2-i+2}\} \) be the subspace \( \text{Span}(\Lambda_i \setminus \Lambda_i) \) of \( W \), and let \( \overline{W}_i = \text{Span}(\Lambda_i) \). For each \((i, j) \in [0, 1] \times [0, 1, 2] \), let \( p_j^{(i)} = [v_j \otimes w_j^{(i)} \wedge w_j^{(2-i)}] \in \text{Span}(\Lambda_i) \).
Suppose that \( \ell \geq 2 \). Then since
\[
\left( V \otimes \bigwedge^2 W_0 \right) \cap \left( V \otimes \bigwedge^2 W_1 \right) = V \otimes \bigwedge^2 \text{Span}(\Lambda_0 \cap \Lambda_1),
\]
we obtain \( \dim \left( V \otimes \bigwedge^2 W_0 \right) \cap \left( V \otimes \bigwedge^2 W_1 \right) = 2 \cdot 3^{(4(\ell - 1) + 3)/2} - 3^{(4(\ell - 2) + 3)/2} \). Therefore
\[
\dim A_2(\ell) \leq 2 \cdot 3^{(4(\ell - 1) + 3)/2} - 3^{(4(\ell - 2) + 3)/2} + 2 \cdot 3 \cdot 8 = 3^{4\ell + 3}/2
\]
with equality being what is generally expected.

**Remark 6.1.** Let \((i, j) \in \{0, 1\} \times \{0, 1, 2\} \). For simplicity, let us denote \( T_{\rho_j} \text{Seg}(\mathbb{P}V \times \mathbb{G}(1, \mathbb{P}W_i)) \) of \( \mathbb{P}(V \otimes \bigwedge^2 W) \) by \( T_{i,j} \), \( \bigcup_{(i, j) \in \{0, 1\} \times \{0, 1, 2\}} T_{i,j} \) by \( T \), and \( \mathbb{P}(V \otimes \bigwedge^2 W_i) \) by \( L_i \). Then the condition that \( \dim A_2(\ell) = 3^{4\ell + 3}/2 \) is equivalent to the condition that there are no hyperplanes of \( \mathbb{P}(V \otimes \bigwedge^2 W) \) containing the union of the \( L_i \)'s and the \( T_{i,j} \)'s. In other words, to show \( \dim A_2(\ell) = 3^{4\ell + 3}/2 \), it is equivalent to showing \( \dim \mathcal{H}(\mathbb{P}(V \otimes \bigwedge^2 W), \mathcal{J}_L) = 0 \).

**Proposition 6.2.** \( \dim A_2(\ell) = 3^{4\ell + 3}/2 \) for each \( \ell \geq 2 \).

**Proof.** For each \( i \in \{0, 1\} \), let \( \Xi_i = \{4(\ell - i)+1, \ldots, 4(\ell - i)+2\} \). Since \( W_i = \text{Span}(\Lambda \setminus \Lambda_i) \), \( \sum_{i=0}^1 V \otimes W_i \) is spanned by \( \{e_\alpha \otimes f_\beta \wedge f_\gamma \mid \alpha \in \{0, 1, 2\}, (\beta, \gamma) \notin \Xi_0 \times \Xi_1 \text{ with } \beta < \gamma\} \).

By semi-continuity, it suffices to show that there are \( p_j^{(i)} \in \text{Seg}(\mathbb{P}V \times \mathbb{G}(1, \mathbb{P}W_i)) \) such that \( A_2(\ell) = V \otimes \bigwedge^2 W \). Consider the three points
\[
\begin{align*}
p_0^{(0)} &= [(e_0 - e_1) \otimes (f_{4(\ell - 1)} + f_{4(\ell - 2)}) \wedge (f_{4(\ell - 1)} + f_{4(\ell - 2)})] \\
p_0^{(1)} &= [(e_1 - e_2) \otimes (f_{4(\ell - 1)} + 2f_{4(\ell - 1)}) \wedge (f_{4(\ell - 1)} + 2f_{4(\ell - 1)})] \\
p_0^{(2)} &= [(e_2 - e_0) \otimes (f_{4(\ell - 1)} + 3f_{4(\ell - 1)}) \wedge (f_{4(\ell - 1)} + 3f_{4(\ell - 1)})]
\end{align*}
\]
in \( \dim(\mathbb{P}V \times \mathfrak{G}(1, \mathbb{P}W_0)) \) and the three points
\[
\begin{align*}
p_1^{(0)} &= [(e_0 + e_1) \otimes (f_4 + f_5) \wedge (f_6 + f_7 + f_8 + f_9)]; \\
p_1^{(1)} &= [(e_0 + e_1) \otimes (f_4 + f_5 + 2f_6) \wedge (f_7 + f_8 + 2f_9)]; \\
p_1^{(2)} &= [(e_0 + e_1) \otimes (f_4 + f_5 + 3f_6) \wedge (f_7 + f_8 + 3f_9)];
\end{align*}
\]
in \( \dim(\mathbb{P}V \times \mathfrak{G}(1, \mathbb{P}W_1)) \). Some tedious manipulation shows that \( e_0 \otimes f_\beta \wedge f_\gamma \) lies in \( A_2(\ell) \) for each \( (\alpha, \beta, \gamma) \in \{0, 1, 2\} \times \Xi_0 \times \Xi_1 \). Thus we completed the proof. \( \square \)

Let \( \ell \geq 2 \). For each \( i \in \{0, 1, 2\} \), let \( p_j = [v_i \otimes w_{j,0} \wedge w_{j,1}] \in \dim(\mathbb{P}V \times \mathfrak{G}(1, \mathbb{P}W)) \) be generic and, for each \( i \in \{3, \ldots, 3\ell + 1\} \), let \( p_i = [v_i \otimes w_{i,0} \wedge w_{i,1}] \in \dim(\mathbb{P}V \times \mathfrak{G}(1, \mathbb{P}W_0)) \) be generic. Consider the subspace \( A_1(\ell) \) of \( V \otimes \wedge^2 W \) given by
\[
A_1(\ell) = V \otimes \bigwedge^2 W_0 + \sum_{i=0}^{3\ell+1} T_p, \dim(\mathbb{P}V \times \mathfrak{G}(1, \mathbb{P}W)).
\]
Then it follows from (6.1) that
\[
\dim A_1(\ell) \leq \min\left(3\left(\frac{3\ell+3}{2}\right)\right) + 8(3\ell - 2) + 3[2 + 2(4\ell + 2 - 1)], 3\left(\frac{3\ell+3}{2}\right))
\]
\[
= 3\left(\frac{3\ell+3}{2}\right)
\]
\[
= \dim V \otimes \bigwedge^2 W.
\]

**Lemma 6.3.** \( \dim A_1(\ell) = 3\left(\frac{4\ell+3}{2}\right) \) for each \( \ell \geq 2 \).

**Proof.** The proof is by induction on \( \ell \). We first prove that \( \dim A_1(2) = 66 \). To do so, it is sufficient to find \( p_0, \ldots, p_2 \in \dim(\mathbb{P}V \times \mathfrak{G}(1, \mathbb{P}W)) \) and \( p_3, \ldots, p_{3\ell+1} \) in \( \dim(\mathbb{P}V \times \mathfrak{G}(1, \mathbb{P}W_0)) \) such that
\[
V \otimes \bigwedge^2 W_0 + \sum_{i=0}^{3\ell+1} T_p, \dim(\mathbb{P}V \times \mathfrak{G}(1, \mathbb{P}W)) = V \otimes \bigwedge^2 W.
\]
Consider the three points of \( \dim(\mathbb{P}V \times \mathfrak{G}(1, \mathbb{P}W)) \)
\[
\begin{align*}
p_0 &= [(e_0 + e_1) \otimes (f_0 + f_1 + f_2) \wedge (f_3 + f_4 + f_5)]; \\
p_1 &= [(e_0 + e_1) \otimes (f_0 + f_1 + 2f_2 + f_3) \wedge (f_4 + f_5 + 2f_6)]; \\
p_2 &= [(e_0 + e_1) \otimes (f_0 + f_1 + 3f_2 + f_3) \wedge (f_4 + f_5 + 3f_6)];
\end{align*}
\]
and the five points of \( \dim(\mathbb{P}V \times \mathfrak{G}(1, \mathbb{P}W_0)) \)
\[
\begin{align*}
p_3 &= [(e_0 - e_1) \otimes (f_0 + f_1 + f_2 + f_3) \wedge (f_4 + f_5 + f_6)]; \\
p_4 &= [(e_0 - e_1) \otimes (f_0 + f_1 + 2f_2 + f_3) \wedge (f_4 + f_5 + 2f_6)]; \\
p_5 &= [(e_0 - e_1) \otimes (f_0 + f_1 + 3f_2 + f_3) \wedge (f_4 + f_5 + 3f_6)]; \\
p_6 &= [e_0 \otimes (f_0 + f_1 + f_2 + f_3) \wedge (f_4 + f_5 + f_6)]; \\
p_7 &= [e_1 \otimes (f_0 + f_1 + f_2 + f_3) \wedge (f_4 + f_5 + f_6)].
\end{align*}
\]
Then it is straightforward to show that (6.2) holds for these \( p_0, \ldots, p_{10} \).

Next assume that \( \dim A_1(\ell - 1) = 3\left(\frac{4\ell-1+3}{2}\right) \) for some \( \ell \geq 2 \). We show \( \dim A_1(\ell) = 3\left(\frac{4\ell+3}{2}\right) \) under this assumption. Let \( p_0, \ldots, p_2 \in \dim(\mathbb{P}V \times \mathfrak{G}(1, \mathbb{P}W)) \) and let \( p_3, \ldots, p_{3\ell+1} \in \dim(\mathbb{P}V \times \mathfrak{G}(1, \mathbb{P}W_0)) \). Let \( L_0 = \mathbb{P}(V \otimes \bigwedge^2 W_0) \). For each \( i \in \{0, \ldots, 3\ell+1\} \), let \( T_i = T_p, \dim(\mathbb{P}V \times \mathfrak{G}(1, \mathbb{P}W)) \) and let \( T = \bigcup_{i=0}^{3\ell+1} T_i \). As in Remark 6.1, the condition that (6.2)
follows that \( \sigma \sum \) the proof of Lemma 6.3. Then a straightforward calculation shows that \( \sum \) dimension at least \( \sum \) at most \( G \) Let \( \wedge \) \( V \otimes p \) the choice of \( \sum \) from which it follows that (6.3) equals the vector space sum of \( p \)'s.

Let \( L_1 = \mathbb{P}(V \otimes \bigwedge^2 W) \). Then we have an exact sequence
\[
0 \to J_{L_0,L_1,T}(1) \to J_{L_0,T}(1) \to J_{L_0,L_1,T}(1) \to 0.
\]
Taking cohomology, we obtain
\[
\dim H^0(\mathbb{P}(V \otimes \bigwedge^2 W), J_{L_0,T}(1)) \leq \dim H^0(\mathbb{P}(V \otimes \bigwedge^2 W), J_{L_0,L_1,T}(1)) + \dim H^0(L_1, J_{L_0,T}(1)) \text{.}
\]
By the induction hypothesis, \( \dim H^0(L_1, J_{L_0,L_1,T}(1)) = 0 \). Thus it is sufficient to show that \( \dim H^0(\mathbb{P}(V \otimes \bigwedge^2 W), J_{L_0,L_1,T}(1)) = 0 \). Note that the latter condition is equivalent to the condition that
\[
V \otimes \bigwedge^2 W_0 + V \otimes \bigwedge^2 W_1 + \sum_{i=0}^{3\ell+1} T_{i, \text{ Seg}}(\mathbb{P}(V \times G(1, PW))) = V \otimes \bigwedge^2 W \text{.}
\]
Recall that if \( i \in \{0, 1\} \) and if \( p = [v \otimes w_0 \wedge w_1] \in \text{ Seg}(\mathbb{P}(V \times G(1, PW))) \), then
\[
\overline{T}_p \text{ Seg}(\mathbb{P}(V \times G(1, PW))) \equiv v \otimes \overline{W}_1 \wedge w_1 + v \otimes w_0 \wedge \overline{W}_1 \pmod{V \otimes \bigwedge^2 W_1} \text{.}
\]
This means that if \( p \in \text{ Seg}(\mathbb{P}(V \times G(1, PW))) \), then
\[
\overline{T}_p \text{ Seg}(\mathbb{P}(V \times G(1, PW))) \equiv \emptyset \pmod{V \otimes \bigwedge^2 W_0 + V \otimes \bigwedge^2 W_1} \text{,}
\]
from which it follows that (6.3) equals the vector space sum of \( V \otimes \bigwedge^2 W_0 + \sum_{i=0}^{3\ell+1} T_{i, \text{ Seg}}(\mathbb{P}(V \times G(1, PW))) \) and \( V \otimes \bigwedge^2 W_1 + \sum_{i=0}^{2\ell+3} T_{i, \text{ Seg}}(\mathbb{P}(V \times G(1, PW))) \). Thus, by the choice of \( p_0, \ldots, p_5 \), (6.3) can be thought of as \( A_2(\ell) \), and hence it coincides with \( V \otimes \bigwedge^2 W \) by Proposition 6.2. Therefore, we completed the proof. \( \square \)

**Theorem 6.4.** For each non-negative integer \( \ell \), \( \sigma_{3\ell+2}(\text{ Seg}(\mathbb{P}(V \times G(1, 4\ell + 2))) \subset \mathbb{P}(V \otimes \bigwedge^2 W) \) is a hypersurface.

**Proof.** Let \( V \) be an three-dimensional vector space over \( k \) with basis \( \{e_0, \ldots, e_2\} \) and let \( W \) be an \((n+1)\)-dimensional vector space over \( k \) with basis \( \{f_0, \ldots, f_{4\ell+2}\} \). If \( \ell = 0 \), then \( G(1, 2) \simeq \mathbb{P}^2 \) and \( \text{ Seg}(\mathbb{P}^2 \times \mathbb{P}^2) \) is known to be a hypersurface in \( \mathbb{P}^8 \). So we may assume that \( \ell \geq 1 \).

In Theorem 4.2 we showed that the dimension of \( \sigma_{3\ell+2}(\text{ Seg}(\mathbb{P}(V \times G(1, 4\ell + 2))) \) is at most \( 3^\ell \left( 2^{4\ell+3} \right) - 2 \). Thus it is sufficient to show that \( \sigma_{3\ell+2}(\text{ Seg}(\mathbb{P}(V \times G(1, 4\ell + 2))) \) has dimension at least \( 3^\ell \left( 2^{4\ell+3} \right) - 2 \). We prove this by induction on \( \ell \geq 1 \).

First suppose that \( \ell = 1 \). Let \( p_5, \ldots, p_7 \) be the points of \( \text{ Seg}(\mathbb{P}^2 \times G(1, \mathbb{P}^6)) \) given in the proof of Lemma 6.3. Then a straightforward calculation shows that \( \sum_{i=0}^{3\ell+1} \overline{T}_{p_i} \text{ Seg}(\mathbb{P}^2 \times G(1, \mathbb{P}^6)) \) has dimension 62. This means that if \( q_1, \ldots, q_5 \in \text{ Seg}(\mathbb{P}^2 \times G(1, \mathbb{P}^6)) \) are generic, then \( \sum_{i=1}^{5} \overline{T}_{q_i} \text{ Seg}(\mathbb{P}^2 \times G(1, \mathbb{P}^6)) \) has at least dimension 62. Thus, by Terracini’s lemma, it follows that \( \sigma_5(\text{ Seg}(\mathbb{P}^2 \times G(1, \mathbb{P}^6)) \) has dimension at least 61.
Next assume that the dimension of $\sigma_{3(l-1)+2}(\text{Seg}(\mathbb{P}^2 \times \mathbb{G}(1, 4\ell - 1 + 2)))$ is at least $3^{2(\ell-1)+3} - 2$. In other words, if $q_1, \ldots, q_{3l(\ell-1)+2} \in \text{Seg}(\mathbb{P}V \times \mathbb{G}(1, 4\ell - 1 + 2))$, then

$$\sum_{i=0}^{3l(\ell-1)+1} T_q \text{ Seg}(\mathbb{P}^2 \times \mathbb{G}(1, 4\ell - 1 + 2))$$

has dimension at least $3^{2(\ell-1)+3} - 1$.

Let $W_0 = \text{Span}(f_0, \ldots, f_{d(\ell-1)+2})$. Let $q_0, \ldots, q_{3l+1} \in \text{Seg}(\mathbb{P}V \times \mathbb{G}(1, \mathbb{P}W))$. Assume that $q_0, \ldots, q_{3l+1}$ are generic in $\text{Seg}(\mathbb{P}V \times \mathbb{G}(1, \mathbb{P}W))$, but $q_{3l-1}, q_{3l}, q_{3l+1}$ are generic in $\text{Seg}(\mathbb{P}V \times \mathbb{G}(1, \mathbb{P}W))$. Let $T = \bigcup_{i=0}^{3l+1} T_q \text{ Seg}(\mathbb{P}V \times \mathbb{G}(1, \mathbb{P}W))$ and let $L_0 = \mathbb{P}(V \otimes W_0)$. Then we obtain an exact sequence

$$0 \to J_{T_{\ell_{0,T}}}(1) \to J_T(1) \to J_{T\cap L_{0,T_{0}}}(1) \to 0.$$

Taking cohomology yields

$$\dim H^0(\mathbb{P}(V \otimes \bigwedge^2 W), J_T(1))$$

$$\leq \dim H^0(\mathbb{P}(V \otimes \bigwedge^2 W), J_{T_{\ell_{0,T}}}(1)) + \dim H^0(L_0, J_{T\cap L_{0,T_{0}}}(1)).$$

By the induction hypothesis, we have $\dim H^0(L_0, J_{T\cap L_{0,T_{0}}}(1)) = 1$. Lemma 6.3 implies that $\dim H^0(\mathbb{P}(V \otimes \bigwedge^2 W), J_{T_{\ell_{0,T}}}(1)) = 0$. Thus we obtain

$$\dim H^0(\mathbb{P}(V \otimes \bigwedge^2 W), J_T(1)) \leq 1.$$

This means that if $p_0, \ldots, p_{3l+1} \in \text{Seg}(\mathbb{P}V \times \mathbb{G}(1, \mathbb{P}W))$ are generic, then $\sum_{i=0}^{3l+1} T_q \text{ Seg}(\mathbb{P}V \times \mathbb{G}(1, \mathbb{P}W))$ has dimension at least $3^{2(\ell-1)+3} - 1$, and hence it follows from Terracini’s lemma that $\dim \sigma_{3l+2} \text{ Seg}(\mathbb{P}V \times \mathbb{G}(1, \mathbb{P}W)) \geq 3^{2(\ell-1)+3} - 2$, as required. 

Among the defective secant varieties of Segre-Grassmann varieties we found, the following are hypersurfaces:

(i) $\sigma_s(\text{Seg}(\mathbb{P}^m \times \mathbb{G}(k, n))) \subset \mathbb{P}^{(m+1)(\binom{n}{k+1})-1}$, where $\text{Seg}(\mathbb{P}^m \times \mathbb{G}(k, n))$ is unbalanced and $s = \min\{m, (\binom{n}{k+1})-1\}$.

(ii) $\sigma_{3l+2}(\text{Seg}(\mathbb{P}^2 \times \mathbb{G}(1, 4\ell + 2))) \subset \mathbb{P}^{3(\binom{n}{k+1})-1}$ with $\ell \geq 0$.

(iii) $\sigma_{3}(\text{Seg}(\mathbb{P}^2 \times \mathbb{G}(2, 5))) \subset \mathbb{P}^{59}$.

If $\sigma_s(\text{Seg}(\mathbb{P}^m \times \mathbb{G}(k, n)))$ is unbalanced and $s = \min\{m, (\binom{n}{k+1})-1\}$, then

$$\sigma_s(\text{Seg}(\mathbb{P}^m \times \mathbb{G}(k, n))) = \sigma_s\left(\text{Seg}(\mathbb{P}^{m} \times \mathbb{P}^{(\binom{n}{k+1})-1})\right).$$

In particular, the determinant of the generic $(m+1) \times (\binom{n+1}{k+1})$ matrix is an equation for Hypersurface (i).

Let $V$ be a three-dimensional vector space over $\mathbb{k}$ with basis $\{e_0, e_1, e_2\}$ and let $W$ be a $(4\ell + 3)$-dimensional vector space over $\mathbb{k}$ with basis $\{f_0, \ldots, f_{d(\ell+2)}\}$. Let $\varphi = \sum_{i,j,k} x_{ijk} e_i \otimes f_j \wedge f_k$ and let $A_\varphi : V \otimes W^* \to \bigwedge^2 V \otimes \mathbb{W}$ be the linear transformation as defined in Section 4. Then it is plausible to guess that $\text{det}(A_\varphi)$ defines Hypersurface (ii) unless $\ell = 0$ (if $\ell = 0$, then the cube root of $\text{det}(A_\varphi)$ is an equation for $\sigma_2(\text{Seg}(\mathbb{P}^2 \times \mathbb{G}(1, 2))) = \sigma_2(\text{Seg}(\mathbb{P}^2 \times (\mathbb{P}^2)^*))$).

Unfortunately, we were unable to find an equation for Hypersurface (iii). We do not even have any plausible guess for such an equation. We thus conclude this paper with the following problem:

**Problem 6.5.** Find an equation for $\sigma_3(\text{Seg}(\mathbb{P}^2 \times \mathbb{G}(2, 5)))$. 
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