ON THE DIMENSIONS OF SECANT VARIETIES OF SEGRE-VERONESE VARIETIES

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Abstract. This paper explores the dimensions of higher secant varieties to Segre-Veronese varieties. The main goal of this paper is to introduce two different inductive techniques. These techniques enable one to reduce the computation of the dimension of the secant variety in a high dimensional case to the computation of the dimensions of secant varieties in low dimensional cases. As an application of these inductive approaches, we will prove non-defectivity of secant varieties of certain two-factor Segre-Veronese varieties. We also use these methods to give a complete classification of defective s^{th} Segre-Veronese varieties for small s. In the final section, we propose a conjecture about defective two-factor Segre-Veronese varieties.

1. Introduction

In many applications, it is natural to represent a collection of data as a multi-indexed list. Alternatively, one can think of the data as a multi-dimensional array. A mathematical framework that includes the study of multi-dimensional arrays is through parameter spaces of tensors.

Every tensor can be written as a linear combination of so-called *decomposable tensors*. A tensor is said to have *rank s* if it can be written as a linear combination of s decomposable tensors (but not fewer). Note that there are higher rank tensors that can be written as the limit of lower rank tensors. A tensor is said to have *border rank s* if it can be expressed as the limit of rank s tensors, but not as the limit of rank $s-1$ tensors. An interesting question is "Given a positive integer s, what is the dimension of the parameter space of tensors with border rank at most s?". In the following few paragraphs, we will formulate this problem as a classical problem in algebraic geometry.

Let k be a positive integer. For each $i \in \{1, \dots, k\}$, let V_i be a vector space of dimension $n_i + 1$ over $\mathbb{C}$, $n_1 \leq \cdots \leq n_k$. The collection of decomposable tensors can be embedded into the N-dimensional vector space $\bigotimes_{i=1}^k V_i$, where $N = \prod_{i=1}^k (n_i + 1)$. Projectivizing to account for the effect of scalars, we have a *Segre map* $\prod_{i=1}^k \mathbb{P}(V_i) \to \mathbb{P}\left(\bigotimes_{i=1}^k V_i\right)$. The image of this map, denoted X, is called the *Segre variety*.

A *secant $(s-1)$-plane* to X is a linear subspace that passes through s linearly independent points of X. Each point on the secant $(s-1)$-plane is a linear combination of s points on X and can be identified with a tensor which is a linear combination of s fixed decomposable tensors. The Zariski closure of the set of all

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points which lie on a secant $(s-1)$-plane, i.e., the set of all tensors that can be written as the sum of s decomposable tensors, is called the s^{th} *secant variety* of X and denoted by $\sigma_s(X)$. The variety $\sigma_s(X)$ parameterizes tensors with border rank at most s. Thus the aforementioned question is equivalent to the question about "What is the dimension of $\sigma_s(X)$?".

Since $\sigma_s(X) \subset \mathbb{P}^{N-1}$ is the closure of the union of secant $(s-1)$-planes to X, the following inequality holds:

$$\dim \sigma_s(X) \leq \min \left\{ N-1, s\left(1+\sum_{i=1}^{k} n_i\right) - 1 \right\}.$$

We say that $\sigma_s(X)$ has the *expected dimension* if the equality holds. The Segre variety X has a *defective* s^{th} *secant variety* if $\sigma_s(X)$ does not have the expected dimension. In particular, X is called *defective* if X has a defective s^{th} secant variety for some s. For example, if $k=2$, then X corresponds to the parameter space of rank one $(n_1+1)\times(n_2+1)$ matrices, and the points of $\sigma_s(X)$ correspond to $(n_1+1)\times(n_2+1)$ matrices that can be written as the sum of s (or fewer) rank one matrices of the same size. Thus most of secant varieties to two-factor Segre varieties are defective, because points which lie on a secant (n_1-1)-plane lie on the algebraic set defined by the ideal of maximal minors (the same argument proves that X is not defective if $n_1 = 1$). On the other hand, there are only a few families of defective Segre varieties known to exist for $k \geq 3$. It is therefore desirable to classify defective Segre varieties.

There are other categories of tensors such as symmetric tensors, alternating tensors and hybrids of regular tensors and symmetric tensors. Those tensors also arise very naturally throughout physics, computer science, engineering as well as mathematics.

The concepts of rank and border rank of regular tensors can be extended to tensors in other categories. The geometry of decomposable tensors in each of these categories can be analogously exploited: Veronese varieties, Grassmann varieties and Segre-Veronese varieties can be thought of as parameter spaces of decomposable symmetric tensors, decomposable alternating tensors and decomposable mixed regular and symmetric tensors respectively, and questions about rank of tensors in each category are related to questions about secant varieties of the corresponding varieties.

A well known classification of the defective Veronese varieties was completed in a series of papers by Alexander and Hirschowitz [6]. There are corresponding conjecturally complete lists of defective Segre varieties [4] and Grassmann varieties [9]. Defective secant varieties of Segre-Veronese varieties are, however, less well-understood, although considerable efforts have been already made to complete the list of such varieties (see for example, [16], [14], [7], [12], [23], [5]). Even the classification of defective two-factor Segre-Veronese varieties is still far from complete.

One of the main goals of this paper is to provide an array of tools to study secant varieties of Segre-Veronese varieties. In order to classify defective Segre-Veronese varieties, a crucial step is to prove the existence of a large family of non-defective such varieties. A powerful tool to establish non-defectivity of large classes of Segre-Veronese varieties is the inductive approach based on specialization techniques, which consist in placing a certain number of points on a chosen divisor. For a given

$\mathbf{n} = (n_1, \ldots, n_k) \in \mathbb{N}^k$, we denote $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ by $\mathbb{P}^{\mathbf{n}}$. Let $X_{\mathbf{n}}^{\mathbf{a}}$ be the Segre-Veronese variety obtained by embedding $\mathbb{P}^{\mathbf{n}}$ in $\mathbb{P}^{\prod_{i=1}^{k}\binom{n_i+a_i}{a_i}-1}$ by the morphism given by $\mathcal{O}(\mathbf{a})$ with $\mathbf{a} = (a_1, \ldots, a_k) \in \mathbb{N}^k$. Note that the problem of determining the dimension of $\sigma_s(X_{\mathbf{n}}^{\mathbf{a}})$ is equivalent to the problem of determining the value of the Hilbert function $h_{\mathbb{P}^{\mathbf{n}}}(Z, \cdot)$ of a collection Z of s general double points in $\mathbb{P}^{\mathbf{n}}$ at $\mathbf{a}$, i.e.,

$$h_{\mathbb{P}^{\mathbf{n}}}(Z, \mathbf{a}) = \dim H^0(\mathbb{P}^{\mathbf{n}}, \mathcal{O}(\mathbf{a})) - \dim H^0(\mathbb{P}^{\mathbf{n}}, \mathcal{I}_Z(\mathbf{a})).$$

Suppose that $a_1 \geq 2$. Denote by $\mathbf{n}'$ and $\mathbf{a}'$ the k-tuples $(n_1 - 1, n_2, \ldots, n_k)$ and $(a_1 - 1, a_2, \ldots, a_k)$ respectively. Given a $\mathbb{P}^{\mathbf{n}'} \subset \mathbb{P}^{\mathbf{n}}$, we have a short exact sequence

$$0 \to \mathcal{I}_{\widetilde{Z}}(\mathbf{a}') \to \mathcal{I}_Z(\mathbf{a}) \to \mathcal{I}_{Z\cap\mathbb{P}^{\mathbf{n}'},\mathbb{P}^{\mathbf{n}'}}(\mathbf{a}) \to 0,$$

where $\widetilde{Z}$ is the residual scheme of Z with respect to $\mathbb{P}^{\mathbf{n}'}$ and $Z \cap \mathbb{P}^{\mathbf{n}'}$ is the trace of Z on the hyperplane. This short exact sequence gives rise to the so-called *Castelnuovo inequality*

$$h_{\mathbb{P}^{\mathbf{n}}}(Z, \mathbf{a}) \geq h_{\mathbb{P}^{\mathbf{n}}}(\widetilde{Z}, \mathbf{a}') + h_{\mathbb{P}^{\mathbf{n}'}}(Z \cap \mathbb{P}^{\mathbf{n}'}, \mathbf{a}).$$

Thus, we can conclude that

(a) if $h_{\mathbb{P}^{\mathbf{n}}}(\widetilde{Z}, \mathbf{a}')$ and $h_{\mathbb{P}^{\mathbf{n}'}}(Z \cap \mathbb{P}^{\mathbf{n}'}, \mathbf{a}')$ are the expected values and

(b) if the degrees of $\widetilde{Z}$ and $Z \cap \mathbb{P}^{\mathbf{n}'}$ are both less than or both greater than $\dim H^0(\mathbb{P}^{\mathbf{n}}, \mathcal{O}(\mathbf{a}'))$ and $\dim H^0(\mathbb{P}^{\mathbf{n}'}, \mathcal{O}(\mathbf{a}))$ respectively,

then $h_{\mathbb{P}^{\mathbf{n}}}(Z, \mathbf{a})$ is also the expected value. By semicontinuity, the Hilbert function of a general collection of s double points in $\mathbb{P}^{\mathbf{n}}$ has the expected value at $\mathbf{a}$.

The problem is, however, that it may or may not be possible to arrange that Condition (b) is satisfied. In Section 2 we generalize the *méthode d'Horace différentielle* of Alexander and Hirschowitz [6] to give a way around this numerical obstacle. More precisely, we will prove the following theorem:

Theorem 1.1. *Let $a_1 \geq 3$. Let $\mathbf{n}' = (n_1-1, n_2, \ldots, n_k)$, let $\mathbf{a}' = (a_1-1, a_2, \ldots, a_k)$, and let $\mathbf{a}'' = (a_1 - 2, a_2, \ldots, a_k)$. For a given positive integer s, let s' and ϵ be the quotient and remainder when dividing $s\left(1 + \sum_{i=1}^{k} n_i\right) - \binom{n_1+a_1-1}{a_1-1}\prod_{i=2}^{k}\binom{n_i+a_i}{a_i}$ by $\sum_{i=1}^{k} n_i$. Suppose that $s' \geq \varepsilon$. If $\sigma_{s'}(X_{\mathbf{n}',\mathbf{a}})$, $\sigma_{s-s'}(X_{\mathbf{n},\mathbf{a}'})$, and $\sigma_{s-s'-\epsilon}(X_{\mathbf{n},\mathbf{a}''})$ have the expected dimension and if*

$$(1) \qquad (s - s' - \epsilon)\left(1 + \sum_{i=1}^{k} n_i\right) \geq \binom{n_1 + a_1 - 2}{a_1 - 2}\prod_{i=2}^{k}\binom{n_i + a_i}{a_i},$$

then $\sigma_s(X_{\mathbf{n},\mathbf{a}})$ also has the expected dimension.

This theorem enables one to check whether or not $\sigma_s(X_{\mathbf{n},\mathbf{a}})$ has the expected dimension by induction on $\mathbf{n}$ and $\mathbf{a}$. It cannot however be applied to $\sigma_s(X_{\mathbf{n},\mathbf{a}})$ if $\mathbf{a}$ is small. The theorem requires that one of the a_i's is at least 3, so one cannot use it when every a_i is less than or equal to two. In addition, if at least one of the degrees is 1, it is frequent that Inequality (1) does not hold. In Section 2, we therefore develop a different inductive approach for computing the dimensions of secant varieties of such Segre-Veronese varieties. This approach allows one to places a certain number of points not only on a hypersurface, but also on a subvariety (see Theorem 2.13 for a more precise statement). Note that a similar approach was successfully applied to study secant varieties of Segre varieties in [3].

In order to apply these inductive approaches, we need some initial cases regarding either dimensions or degrees. The class of secant varieties of two-factor Segre-Veronese varieties can be viewed as one of such initial cases. In Section 3, we will study secant varieties of such Segre-Veronese varieties. The main goal of this section is to prove the following theorem:

Theorem 1.2. *Let $n, a \geq 1$, $b \geq 3$, $\mathbf{n} = (n, 1)$ and $\mathbf{a} = (a, b)$. Then $X_{\mathbf{n},\mathbf{a}}$ is not defective except if $(n, a, b) = (n, 2, 2k)$.*

As an application of Theorem 1.1 together with Theorem 1.2, we prove also the following theorem:

Theorem 1.3. *Suppose that $X_{\mathbf{n},\mathbf{a}}$ is not defective for every $\mathbf{n}$ and for $\mathbf{a} = (3, 3)$, $(3, 4)$ and $(4, 4)$. Then $X_{\mathbf{n},\mathbf{a}}$ is not defective for every $\mathbf{n}$ and for every $\mathbf{a} = (a, b)$ such that $a, b \geq 3$.*

As we shall see in Section 2, using a randomized algorithm which employs Terracini's lemma, we can compute the dimension of $\sigma_s(X_{\mathbf{n},\mathbf{a}})$ for a given $s \in \mathbb{N}$ and for given $\mathbf{n}, \mathbf{a} \in \mathbb{N}^k$. Based on our experiments using this randomized algorithm, we anticipate that there are no defective Segre-Veronese varieties $X_{\mathbf{n},\mathbf{a}}$ for every $\mathbf{n}$ if $\mathbf{a} = (3, 3)$, $(3, 4)$ or $(4, 4)$. Thus Theorem 1.3 suggests the following conjecture:

Conjecture 1.4. *Let $\mathbf{n}$ and $\mathbf{a}$ be pairs of positive integers. If $\mathbf{a} \geq (3, 3)$, there are no defective two-factor Segre-Veronese varieties $X_{\mathbf{n},\mathbf{a}}$ for all $\mathbf{n} \in \mathbb{N}^2$.*

In Section 4, we apply the inductive procedures developed in Section 2 to classify all the defective s^{th} secant varieties of Segre-Veronese varieties for each $s \in \{2, 3, 4\}$.

Section 5 provides a conjecturally complete list of defective secant varieties of two-factor Segre-Veronese varieties. In addition to evidence provided by our theorems, further evidence in support of the conjecture was obtained via the computational experiments we carried out with `Macaulay2`, a computer algebra system developed by Dan Grayson and Mike Stillman [22].

2. Inductive techniques

For each $i \in \{1, \ldots, k\}$, let V_i be a $(n_i + 1)$-dimensional vector space over $\mathbb{C}$ and let $\mathbb{P}^{n_i} = \mathbb{P}(V_i)$. Given two k-tuples $\mathbf{n} = (n_1, \cdots, n_k)$ and $\mathbf{m} = (m_1, \cdots, m_k)$, we write $\mathbf{n} \leq \mathbf{m}$ when $n_i \leq m_i$ for any i. Unless otherwise stated, $\mathbf{n}$, $\mathbf{n}'$, $\mathbf{a}$, $\mathbf{a}'$ and $\mathbf{a}''$ denote $(n_1, \cdots, n_k)$, $(n_1 - 1, n_2, \cdots, n_k)$, $(a_1, \cdots, a_k)$, $(a_1 - 1, a_2, \cdots, a_k)$ and $(a_1 - 2, a_2, \cdots, a_k) \in \mathbb{N}^k$ respectively. We write $\mathbb{P}^{\mathbf{n}}$ for $\prod_{i=1}^k \mathbb{P}^{n_i}$ and $X_{\mathbf{n},\mathbf{a}}$ for the Segre-Veronese variety embedded in $\mathbb{P}^{N-1}$ by $\mathcal{O}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{a})$, where $N = \prod_{i=1}^k \binom{n_i+a_i}{a_i}$. Let $N_R = \binom{n_1+a_1-1}{a_1-1} \prod_{i=2}^k \binom{n_i+a_i}{a_i}$ and let $N_T = \binom{n_1+a_1-1}{a_1} \prod_{i=2}^k \binom{n_i+a_i}{a_i}$. Let $R = \mathbb{C}[x_{0,1}, \ldots, x_{n_1,1}, \ldots, x_{0,k}, \ldots, x_{n_k,k}]$ and note that it can be thought of as an $\mathbb{N}^k$-graded ring in the obvious way.

Let $\sigma_s(X_{\mathbf{n},\mathbf{a}})$ be the s^{th} secant variety of $X_{\mathbf{n},\mathbf{a}}$, i.e., the Zariski closure of the union of linear subspaces spanned by s-tuples of points on $X_{\mathbf{n},\mathbf{a}}$. We now explain how to translate the problem of computing the dimension of $\sigma_s(X_{\mathbf{n},\mathbf{a}})$ into a question about the value of the Hilbert function of the ideal of s double points on $\mathbb{P}^{\mathbf{n}}$ at $\mathbf{a}$. Let $\mathbb{T}_p(X_{\mathbf{n},\mathbf{a}})$ be the projective tangent space to $X_{\mathbf{n},\mathbf{a}}$ at a point p. The following well known result describes the tangent space of $\sigma_s(X_{\mathbf{n},\mathbf{a}})$:

Theorem 2.1 ((Terracini's lemma))**.** *Let $p_1, \dots, p_s$ be generic points of $X_{\mathbf{n},\mathbf{a}}$ and let q be a generic point of $\langle p_1, \dots, p_s \rangle$. Then*

$$\mathbb{T}_q[\sigma_s(X_{\mathbf{n},\mathbf{a}})] = \sum_{i=1}^{s} \mathbb{T}_{p_i}(X_{\mathbf{n},\mathbf{a}}),$$

where $\mathbb{T}_q[\sigma_s(X_{\mathbf{n},\mathbf{a}})]$ is the projective tangent space to $\sigma_s(X_{\mathbf{n},\mathbf{a}})$ at $q \in \sigma_s(X_{\mathbf{n},\mathbf{a}})$.

Remark 2.2. Let $\mathbf{n}$ and $\mathbf{a}$ be k-tuples of non-negative integers. Let k be a positive integer. For an $i \in \{1, \dots, k\}$, let V_i be an $(n_i + 1)$-dimensional vector space over $\mathbb{C}$ and let $v_i \in V_i \setminus \{0\}$. Denote by $p \in X_{\mathbf{n},\mathbf{a}}$ the equivalence class containing $v_1^{a_1} \otimes \cdots \otimes v_k^{a_k}$. Then the affine cone over $\mathbb{T}_p(X_{\mathbf{n},\mathbf{a}})$ in $\bigotimes_{i=1}^k S_{a_i} V_i$ is

$$C[\mathbb{T}_p(X_{\mathbf{n},\mathbf{a}})] = \sum_{i=1}^{k} v_1^{a_i} \otimes \cdots \otimes v_i^{a_i - 1} V_i \otimes \cdots \otimes v_k^{a_k}.$$

In particular, $C[\mathbb{T}_p(X_{\mathbf{n},\mathbf{a}})]$ can be represented by a $\left[\sum_{i=1}^k (n_i + 1)\right] \times N$ matrix A_p. Thus Terracini's lemma can be used to estimate the dimension of $\sigma_s(X_{\mathbf{n},\mathbf{a}})$ as follows: First choose randomly s points $p_1, \dots, p_s$ on $X_{\mathbf{n},\mathbf{a}}$. Next, compute the matrix representation A_{p_i} for each $C[\mathbb{T}_{p_i}(X_{\mathbf{n},\mathbf{a}})]$. Let A be the matrix obtained by stacking $A_{p_1}, \dots, A_{p_s}$. It follows from Terracini's lemma that $\dim \sigma_s(X_{\mathbf{n},\mathbf{a}}) \geq \operatorname{rank}(A) - 1$. By semi-continuity, the equality holds if $\operatorname{rank}(A) = \min\left\{1 + \sum_{i=1}^k n_i, N\right\}$, because $\dim \sigma_s(X_{\mathbf{n},\mathbf{a}}) \leq \min\left\{\sum_{i=1}^k n_i, N - 1\right\}$. Finally, we would like to stress that although $\operatorname{rank}(A) \neq \min\left\{1 + \sum_{i=1}^k n_i, N\right\}$ is a strong evidence that $\sigma_s(X_{\mathbf{n},\mathbf{a}})$ is defective, it cannot be used to prove defectivity.

Note that $H^0(\mathbb{P}^{\mathbf{n}}, \mathcal{O}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{a}))$ can be identified with the set of hyperplanes in $\mathbb{P}^N$. Since the condition that a hyperplane $H \subset \mathbb{P}^N$ contains $\mathbb{T}_p(X_{\mathbf{n},\mathbf{a}})$ is equivalent to the condition that H intersects $X_{\mathbf{n},\mathbf{a}}$ in the first infinitesimal neighborhood of p, the elements of $H^0(\mathbb{P}^{\mathbf{n}}, \mathcal{I}_p^2(\mathbf{a}))$ can be viewed as hyperplanes containing $\mathbb{T}_p(X_{\mathbf{n},\mathbf{a}})$. Let Z be a collection of s double points on $\mathbb{P}^{\mathbf{n}}$ and let $\mathcal{I}_Z$ be its ideal sheaf. Terracini's lemma implies that $\dim \sigma_s(X_{\mathbf{n},\mathbf{a}})$ equals to the value of the Hilbert function $h_{\mathbb{P}^{\mathbf{n}}}(Z, \cdot)$ of Z at $\mathbf{a}$, i.e.,

$$h_{\mathbb{P}^{\mathbf{n}}}(Z, \mathbf{a}) = \dim H^0(\mathbb{P}^{\mathbf{n}}, \mathcal{O}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{a})) - \dim H^0(\mathbb{P}^{\mathbf{n}}, \mathcal{I}_Z(\mathbf{a})).$$

To prove that $\sigma_s(X_{\mathbf{n},\mathbf{a}})$ has the expected dimension is equivalent to prove that

$$h_{\mathbb{P}^{\mathbf{n}}}(Z, \mathbf{a}) = \min\left\{ s\left(1 + \sum_{i=1}^{k} n_i\right),\ N \right\}.$$

Definition 2.3. Let $\mathbf{n}, \mathbf{a} \in \mathbb{N}^k$, let s be a non-negative integer and Z a zero-dimensional subscheme of $\mathbb{P}^n$. A triple $(\mathbf{n}; \mathbf{a}; Z)$ is said to be *subabundant* (resp. *superabundant*) if $\deg Z \leq N$ (resp. $\deg Z \geq N$). The triple $(\mathbf{n}; \mathbf{a}; Z)$ is said to be *equiabundant* if it is both subabundant and superabundant. We say that $T(\mathbf{n}; \mathbf{a}; Z)$ is *true* if $h_{\mathbb{P}^{\mathbf{n}}}(Z, .)$ has the expected value at $\mathbf{a}$. If Z is a collection of s double points, we write $T(\mathbf{n}; \mathbf{a}; s)$ instead of $T(\mathbf{n}; \mathbf{a}; Z)$ and $(\mathbf{n}; \mathbf{a}; s)$ instead of $(\mathbf{n}; \mathbf{a}; Z)$. We say that $T(\mathbf{n}; \mathbf{a})$ is true if $T(\mathbf{n}; \mathbf{a}; s)$ is true for every $s \geq 0$.

Assume that $a_1 \geq 2$. Let H be a hypersurface defined by a linear form in $R_{(1,0,\dots,0)}$. For a given zero-dimensional subscheme Z, we denote by $\widetilde{Z}$ the *residual* of Z with respect to H, i.e. the subscheme whose ideal is $\mathcal{I}_Z : \mathcal{I}_H$. The scheme $Z \cap H$ is called the *trace* of Z. From the restriction exact sequence

$$0 \to \mathcal{I}_{\widetilde{Z}}(\mathbf{a}') \to \mathcal{I}_Z(\mathbf{a}) \to \mathcal{I}_{Z\cap H}(\mathbf{a}) \to 0,$$

we easily get the so-called Castelnuovo inequality

$$h_{\mathbb{P}^{\mathbf{n}}}(Z, \mathbf{a}) \geq h_{\mathbb{P}^{\mathbf{n}}}(\widetilde{Z}, \mathbf{a}') + h_{\mathbb{P}^{\mathbf{n}'}}(Z \cap H, \mathbf{a}).$$

From this inequality it is easy to prove the following basic Horace lemma.

Theorem 2.4. *Let $a_1 \geq 2$, let Z be a zero-dimensional subscheme of $\mathbb{P}^{\mathbf{n}}$ and let H be a hyperplane defined by a linear form in $R_{(1,0,\ldots,0)}$.*

(i) *If $h_{\mathbb{P}^{\mathbf{n}}}(\widetilde{Z}, \mathbf{a}')$ and $h_{\mathbb{P}^{\mathbf{n}'}}(Z \cap H, \mathbf{a})$ are the expected value;*
(ii) *if $(\mathbf{n}'; \mathbf{a}; Z \cap H)$ and $(\mathbf{n}; \mathbf{a}'; \widetilde{Z})$ have the same abundancy,*

then $h_{\mathbb{P}^{\mathbf{n}}}(Z, \mathbf{a})$ is also the expected value.

Lemma 2.5. *Assume that $a_1 \geq 2$. Let Z be a subscheme of $\mathbb{P}^{\mathbf{n}}$ and let H a hyperplane defined by a linear form in $R_{(1,0,\ldots,0)}$. Then there exists a collection Φ of u general points in H such that*

$$h_{\mathbb{P}^{\mathbf{n}}}(Z \cup \Phi, \mathbf{a}) = h_{\mathbb{P}^{\mathbf{n}}}(Z, \mathbf{a}) + u$$

if and only if u satisfies

$$h_{\mathbb{P}^{\mathbf{n}}}(Z, \mathbf{a}) + u \leq h_{\mathbb{P}^{\mathbf{n}}}(\widetilde{Z}, \mathbf{a}') + \binom{n_1 - 1 + a_1}{n_1 - 1} \prod_{i=2}^{k} \binom{n_i + a_i}{a_i}. \tag{2}$$

Proof. This lemma is an easy generalization of Lemma 3 in [19]. One can prove our statement exactly in the same way as in [19], and thus we omit the proof. $\square$

In the following example, we show how to combine Theorem 2.4 with Lemma 2.5, in order to reduce computing the dimension of the secant variety of a Segre-Veronese variety to computing the dimensions of secant varieties of smaller Segre-Veronese varieties:

Example 2.6. Let $\mathbf{n} = (1, 1)$ and let $\mathbf{a} = (3, 3)$. Let $p_1, \ldots, p_5 \in \mathbb{P}^{\mathbf{n}}$ and let $Z = \{p_1^2, \ldots, p_5^2\}$. Specialize two points, say p_4 and p_5, to $H = \mathbb{P}^0 \times \mathbb{P}^1 \subset (\mathbb{P}^1)^2$. Then $\widetilde{Z}$ consists of three double points and two simple points; while $Z \cap H$ consists of two double points in H. So both $(1, 1; 2, 3; \widetilde{Z})$ and $(0, 1; 3, 3; Z \cap H) = (1; 4; Z \cap H) = (1; 4; 2)$ are subabundant. It is well known that $T(1; 4; 2)$ is true. We therefore want to prove the truth of $T(1, 1; 2, 3; \widetilde{Z})$.

Note that the inequality

$$\begin{aligned} 11 &= 9 + 2 \\ &= h_{\mathbb{P}^{\mathbf{n}}}(\{p_1^2, p_2^2, p_3^2\}, (2, 3)) + 2 \\ &\leq h_{\mathbb{P}^{\mathbf{n}}}(\{p_1^2, p_2^2, p_3^2\}, (1, 3)) + \binom{1+3}{3} \\ &= 8 + 4 = 12, \end{aligned}$$

holds. Thus, by Lemma 2.5, the expected value of the Hilbert function of $\widetilde{Z}$ at $(2, 3)$ is

$$h_{\mathbb{P}^{\mathbf{n}}}(\widetilde{Z}, (2, 3)) = h_{\mathbb{P}^{\mathbf{n}}}(\{p_1^2, p_2^2, p_3^2\}, (2, 3)) + 2 = 11.$$

Additionally, Theorem 2.1 in [16] implies that $T(1,1;2,3;3)$ and $T(1,1;1,3;3)$ are true. Thus, $T(1,1;2,3;\widetilde{Z})$ is true. Therefore, the truth of $T(\mathbf{n};\mathbf{a};5)$ follows from Theorem 2.4.

As already stated in Section 1, one cannot always arrange that Condition (ii) in Theorem 2.4 is satisfied. We illustrate it in the following example:

Example 2.7. Let $\mathbf{n}=(2,2)$, let $\mathbf{a}=(4,4)$, let $p_1,\dots,p_{45}\in\mathbb{P}^{\mathbf{n}}$ and let $Z=\{p_1^2,\dots,p_{45}^2\}$. To prove the truth of $T(\mathbf{n};\mathbf{a};s)$, we want to specialize a certain number of points among the p_i's, say $p_1,\dots,p_{s'}$, to $H\simeq\mathbb{P}^1\times\mathbb{P}^2\subset(\mathbb{P}^2)^2$ in such a way that $(1,2;\mathbf{a};s')$ and $(\mathbf{n};3,4;\widetilde{Z})$ have the same abundancy. This means that they must be equiabundant, because $(\mathbf{n};\mathbf{a};45)$ is equiabundant. It is not possible, however, to find such an integer s', because $\binom{1+4}{4}\binom{2+4}{4}/(1+2+1)\notin\mathbb{Z}$. Thus one cannot apply Theorem 2.4 to show that $T(\mathbf{n};\mathbf{a};s)$ is true.

One of the main goals of this section is to generalize the differential Horace method introduced by Alexander and Hirschowitz to Segre-Veronese varieties in order to side step numerical obstacles like above.

Given a linear system $\mathcal{D}$ on $\mathbb{P}^{\mathbf{n}}$, we say that a scheme Z is *$\mathcal{D}$-independent* if the value $h_{\mathbb{P}^{\mathbf{n}}}(Z,\mathcal{D})=\dim H^0(\mathbb{P}^{\mathbf{n}},\mathcal{D})-\dim H^0(\mathbb{P}^{\mathbf{n}},\mathcal{I}_Z\otimes\mathcal{D})$ equals the degree of the scheme Z. The following lemma is also due to Chandler (see [11, Lemma 6.1] for a detailed proof):

Lemma 2.8. *Let $Z\subset\mathbb{P}^{\mathbf{n}}$ be a zero-dimensional scheme contained in a finite collection of double points and let $\mathcal{D}$ be a linear system on $\mathbb{P}^{\mathbf{n}}$. Then Z is $\mathcal{D}$-independent if and only if every curvilinear subscheme ζ of Z is $\mathcal{D}$-independent.*

We are now able to prove the *méthode d'Horace différentielle* for Segre-Veronese varieties.

Theorem 2.9. *Let $a_1\geq 3$. For a given non-negative integer s, let s' and ε be the quotient and remainder in the division of $s\left(1+\sum_{i=1}^k n_i\right)-N_R$ by $\sum_{i=1}^k n_i$. Suppose that $s'\geq\varepsilon$. If $T(\mathbf{n}';\mathbf{a};s')$, $T(\mathbf{n};\mathbf{a}';s-s')$ and $T(\mathbf{n};\mathbf{a}'';s-s'-\varepsilon)$ are all true and if $(\mathbf{n};\mathbf{a}'';s-s'-\varepsilon)$ is superabundant, then $T(\mathbf{n};\mathbf{a};s)$ is also true.*

Proof. Here we only focus on the case when $(\mathbf{n};\mathbf{a};s)$ is subabundant, because the remaining case can be proved in a similar manner.

Step 1. By assumption, $N_R=\left(1+\sum_{i=1}^k n_i\right)(s-s')-\varepsilon+s'$, and since $s'>\varepsilon$ we have that $(\mathbf{n};\mathbf{a}';s-s')$ is subabundant. This implies that since $T(\mathbf{n};\mathbf{a}';s-s')$ holds by induction, then the Hilbert function $h_{\mathbb{P}^{\mathbf{n}}}(Z,\mathbf{a}')$ has the expected value for any subscheme Z of a collection of $s-s'$ general double points.

Now choose a hyperplane H defined by a linear form in $R_{(1,0,\dots,0)}$. Let $\Gamma=\{\gamma^1,\dots,\gamma^\varepsilon\}$ be a collection of ε general points contained in H and Σ a collection of $s-s'-\varepsilon$ points not contained in H. Let $Z=\Gamma^2_{|H}\cup\Sigma^2$. Then from what we say above it follows

$$h_{\mathbb{P}^{\mathbf{n}}}(Z,\mathbf{a}')=\min\left\{\left(1+\sum_{i=1}^k n_i\right)(s-s')-\varepsilon,N_R\right\}=\left(1+\sum_{i=1}^k n_i\right)(s-s')-\varepsilon.$$

Step 2. Now we want to add to Z a collection Φ of s' simple points contained in H in such a way that

$$h_{\mathbb{P}^{\mathbf{n}}}(Z\cup\Phi,\mathbf{a}')=h_{\mathbb{P}^{\mathbf{n}}}(Z,\mathbf{a}')+s'. \tag{3}$$

By Lemma 2.5 we can do this if

$$h_{\mathbb{P}^{\mathbf{n}}}(Z, \mathbf{a}') + s' \leq h_{\mathbb{P}^{\mathbf{n}}}(\Sigma^2, \mathbf{a}'') + \binom{n_1 + a_1 - 2}{a_1 - 1} \prod_{i=2}^{k} \binom{n_i + a_i}{a_i}.$$

By assumption, $T(\mathbf{n}; \mathbf{a}''; s - s' - \varepsilon)$ is true and $(\mathbf{n}; \mathbf{a}''; s - s' - \varepsilon)$ is superabundant, which implies

$$h_{\mathbb{P}^{\mathbf{n}}}(\Sigma^2, \mathbf{a}'') + \binom{n_1 + a_1 - 2}{a_1 - 1} \prod_{i=2}^{k} \binom{n_i + a_i}{a_i} = N_R.$$

On the other hand, by Step 1 we know that $h_{\mathbb{P}^{\mathbf{n}}}(Z, \mathbf{a}') + s' = N_R$, then Equality (3) follows.

STEP 2. Since $T(\mathbf{n}, \mathbf{a}, s)$ is subabundant, it follows that $s'(\sum_{i=1}^{k} n_i) + \varepsilon \leq N_T$. Then by the inductive hypothesis $T(\mathbf{n}', \mathbf{a}, s')$, it follows that the scheme $(\Gamma \cup \Phi^2_{|H}) \subset H$ has Hilbert function

$$h_{\mathbb{P}^{\mathbf{n}}}(\Gamma \cup \Phi^2_{|H}, \mathbf{a}) = s' \sum_{i=1}^{k} n_i + \varepsilon$$

Now, for $(t_1, \ldots, t_\varepsilon) \in \mathbb{K}^\varepsilon$, choose a flat family of general points $\Delta_{(t_1,\ldots,t_\varepsilon)} = \{\delta^1_{t_1}, \ldots, \delta^\varepsilon_{t_\varepsilon}\} \subseteq \mathbb{P}^{\mathbf{n}}$ and a family of hyperplanes $\{H_{t_1}, \ldots, H_{t_\varepsilon}\}$ defined by linear forms in $R_{(1,0,\ldots,0)}$ such that

- $\delta^i_{t_i} \in H_{t_i}$ for any t_i, and any $i = 1, \ldots, \varepsilon$,
- $\delta^i_{t_i} \notin H$ for any $t_i \neq 0$, and any $i = 1, \ldots, \varepsilon$,
- $H_0 = H$ and $\delta^i_0 = \gamma^i \in H$, for any $i = 1, \ldots, \varepsilon$.

Now let us consider the following schemes:

- $\Delta^2_{(t_1,\ldots,t_\varepsilon)} = \{\delta^1_{t_1}, \ldots, \delta^\varepsilon_{t_\varepsilon}\}^2$, notice that $\Delta^2_{(0,\ldots,0)} = \Gamma^2$;
- Φ^2, where Φ is the collection of the s' points introduced in Step 2;
- Σ^2, the collection of the $s - s' - \varepsilon$ double points introduced in Step 1.

In order to prove $T(\mathbf{n}; \mathbf{a}; s)$ it is enough to prove the following claim.

CLAIM. There exists $(t_1, \ldots, t_\varepsilon)$ such that the scheme $\Delta^2_{(t_1,\ldots,t_\varepsilon)}$ is independent with respect to the linear system $\mathcal{I}_{\Phi^2 \cup \Sigma^2} \otimes \mathcal{O}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{a})$.

Proof of the claim. Assume that the claim is false. Then by Lemma 2.8 for all $(t_1, \ldots, t_\varepsilon)$ there exist pairs $(\delta^i_{t_i}, \eta^i_{t_i})$ for $i = 1, \ldots, \varepsilon$, with $\eta^i_{t_i}$ a curvilinear scheme supported in $\delta^i_{t_i}$ and contained in $\Delta^2_{(t_1,\ldots,t_\varepsilon)}$ such that

$$h_{\mathbb{P}^n}(\Phi^2 \cup \Sigma^2 \cup \eta^1_{t_1} \cup \ldots \cup \eta^\varepsilon_{t_\varepsilon}, \mathbf{a}) < \left(1 + \sum_{i=1}^{k} n_i\right)(s - \varepsilon) + 2\varepsilon. \tag{4}$$

Let η^i_0 be the limit of $\eta^i_{t_i}$, for $i = 1, \ldots, \varepsilon$. Suppose that $\eta^i_0 \not\subset H$ for $i \in F \subseteq \{1, \ldots, \varepsilon\}$ and $\eta^i_0 \subset H$ for $i \in G = \{1, \ldots, \varepsilon\} \setminus F$. Given $t \in \mathbb{K}$, let us denote $Z^F_t = \cup_{i \in F}(\eta^i_t)$ and $Z^G_t = \cup_{i \in G}(\eta^i_t)$. Denote by $\widetilde{\eta}^i_0$ the residual of η^i_0 with respect to H and by f and g the cardinalities respectively of F and G. Then, by (4), we obtain

$$h_{\mathbb{P}^{\mathbf{n}}}(\Phi^2 \cup \Sigma^2 \cup Z^F_0 \cup Z^G_t, \mathbf{a}) < \left(1 + \sum_{i=1}^{k} n_i\right)(s - \varepsilon) + 2\varepsilon. \tag{5}$$

On the other hand, by the semicontinuity of the Hilbert function there exists an open neighborhood O of 0 such that for any $t \in O$

$$h_{\mathbb{P}^{\mathbf{n}}}\left(\Phi \cup \Sigma^2 \cup \eta \cup Z_t^G, \mathbf{a}'\right) \geq h_{\mathbb{P}^{\mathbf{n}}}\left(\Phi \cup \Sigma^2 \cup \eta \cup Z_0^G, \mathbf{a}'\right),$$

where $\eta = \widetilde{\eta}_0^i$. Since $\Phi \cup \Sigma^2 \cup \eta \cup Z_0^G \subseteq \Phi \cup \Sigma^2 \cup \Gamma^2_{|\mathbf{P}^{n-1}}$, by Step 2 we compute

$$h_{\mathbb{P}^{\mathbf{n}}}\left(\Phi \cup \Sigma^2 \cup \eta \cup Z_0^G, \mathbf{a}'\right) = s' + \left(1 + \sum_{i=1}^{k} n_i\right)(s - s' - \varepsilon) + f + 2g.$$

Since $\Phi^2_{|H} \cup \eta$ is a subscheme of $\Phi^2_{|H} \cup \Gamma$, from Step 3 it follows that

$$h_{\mathbb{P}^{\mathbf{n}'}}\left(\Phi^2_{|H} \cup \eta, \mathbf{a}\right) \geq s' \sum_{i=1}^{k} n_i + f$$

Hence for any $0 \neq t \in O$, applying the Castelnuovo inequality to the scheme $\Omega = \Phi^2 \cup \Sigma^2 \cup Z_0^F \cup Z_t^G$, we get

$$\begin{aligned} h_{\mathbb{P}^{\mathbf{n}}}(\Omega, \mathbf{a}) &\geq h_{\mathbb{P}^{\mathbf{n}}}\left(\Phi \cup \Sigma^2 \cup \eta \cup Z_t^G, \mathbf{a}'\right) + h_{\mathbb{P}^{\mathbf{n}'}}\left(\Phi^2_{|H} \cup \eta, \mathbf{a}\right) \\ &\geq s' + \left(1 + \sum_{i=1}^{k} n_i\right)(s - s' - \varepsilon) + f + 2g + s' \sum_{i=1}^{k} n_i + f \\ &= \left(1 + \sum_{i=1}^{k} n_i\right)(s - \varepsilon) + 2\varepsilon, \end{aligned}$$

which contradicts Inequality (5). Thus we completed the proof of the claim. □

Example 2.10. Let $\mathbf{n} = (2, 2)$ and let $\mathbf{a} = (4, 4)$. In Example 2.7, we showed that it is impossible to apply Theorem 2.4 to prove the truth of $T(\mathbf{n}; \mathbf{a}; 45)$. In this example, we illustrate how to reduce $T(\mathbf{n}; \mathbf{a}; 45)$ to computing the dimensions of secant varieties of "smaller" Segre-Veronese varieties using Theorem 2.9.

Let s' and ε be the quotient and remainder when dividing $45(2+2+1) - \binom{5}{2}\binom{6}{2}$ by $2+2$ respectively. Then $s' = 18$ and $\varepsilon = 3$. Thus s' and ε clearly satisfy $s' > \varepsilon$. Since

$$120 = (45 - 18 - 3)(2 + 2 + 1) > \binom{4}{2}\binom{6}{4} = 90,$$

the 5-tuple $(2, 2; 2, 4; 45 - 18 - 3)$ is superabundant. Thus, by Theorem 2.9, one can reduce $T(2, 2; 4, 4; 45)$ to $T(1, 2; 4, 4; 18)$, $T(2, 2; 3, 4; 27)$ and $T(2, 2; 2, 4; 24)$.

Unfortunately, if $k = 2$ and if one of a_i's is 1, then it is often impossible to apply Theorem 2.9. For example, if $(\mathbf{n}; \mathbf{a}; s) = (2, 2; 1, 4; 9)$, then $s' = 3$ and $\varepsilon = 3$. Thus $15 = (9 - 3 - 3)(2 + 2 + 1) < \binom{2+2}{2}(2 + 1) = 18$, and so $(2, 2; 1, 2; 3)$ is not superabundant. Therefore, we cannot reduce $T(2, 2; 1, 4; 9)$ to $T(1, 2; 1, 4; 3)$, $T(2, 2; 1, 3; 6)$ and $T(2, 2; 1, 2; 3)$. Another goal of this section is to provide a different approach to give a way around this kind of problem.

Definition 2.11. Let $a_1 = 1$ and let $\pi : \mathbb{P}^{\mathbf{n}} \to \prod_{i=2}^{k} \mathbb{P}^{n_i}$ be the canonical projection. For each point $p \in \mathbb{P}^{\mathbf{n}}$, let f_p be the double point p^2 restricted to $\pi^{-1}(\pi(p))$. Consider general points $p_1, \ldots, p_s,\ q_1, \ldots, q_t,\ r_1, \ldots, r_v \in \mathbb{P}^{\mathbf{n}}$ and let

$Z = \{p_1^2, \ldots, p_s^2, q_1, \ldots, q_t, f_{r_1}, \ldots, f_{r_v}\}$. We say that the statement $S(\mathbf{n}; \mathbf{a}; s; t; v)$ is true if $T(\mathbf{n}; \mathbf{a}; Z)$ is true, that is, if

$$h_{\mathbb{P}^{\mathbf{n}}}(Z, \mathbf{a}) = \min \left\{ s \left(1 + \sum_{i=1}^{k} n_i \right) + t + v(n_1 + 1),\ N \right\}.$$

We will also write $(\mathbf{n}; \mathbf{a}; s; t; v)$ for $(\mathbf{n}; \mathbf{a}; Z)$.

Remark 2.12. Let $\mathbf{n}$ and $\mathbf{a}$ be k-tuples of non-negative integers. We make the following simple remarks:

(i) $S(\mathbf{n}; \mathbf{a}; s; 0; 0)$ is true if and only if $T(\mathbf{n}; \mathbf{a}; s)$ is true.

(ii) If $(\mathbf{n}; \mathbf{a}; s; t; v)$ is subabundant and if $S(\mathbf{n}; \mathbf{a}; s; t; v)$ is true, then $(\mathbf{n}; \mathbf{a}; s'; t'; v')$ is subabundant and $S(\mathbf{n}; \mathbf{a}; s'; t'; v')$ is true for any choice of s', t' and v' with $s' \le s$, $t' \le t$ and $v' \le v$.

(iii) If $(\mathbf{n}; \mathbf{a}; s; t; v)$ is superabundant and if the statement $S(\mathbf{n}; \mathbf{a}; s; t; v)$ is true, then $(\mathbf{n}; \mathbf{a}; s'; t'; v')$ is superabundant and $S(\mathbf{n}; \mathbf{a}; s'; t'; v')$ is true for any choice of s', t' and v' with $s \le s'$, $t \le t'$ and $v \le v'$. This implies that if $\underline{s} = \left\lfloor \frac{\prod_{i=1}^{k} \binom{n_i + a_i}{a_i}}{1 + \sum_{i=1}^{k} n_i} \right\rfloor$ and $\overline{s} = \left\lceil \frac{\prod_{i=1}^{k} \binom{n_i + a_i}{a_i}}{1 + \sum_{i=1}^{k} n_i} \right\rceil$, then, in order to prove the truth of $T(\mathbf{n}; \mathbf{a})$, it is sufficient to show that $T(\mathbf{n}; \mathbf{a}; s)$ are true for both $s \in \{\underline{s}, \overline{s}\}$.

(iv) The following statements are equivalent and have the same abundancy:
   - $S(0, \mathbf{n}; 1, \mathbf{a}; s; t; v)$.
   - $S(0, \mathbf{n}; 1, \mathbf{a}; s; t + v; 0)$.
   - $S(\mathbf{n}; \mathbf{a}; s; t + v; 0)$.

(v) If $(\mathbf{n}; \mathbf{a}; s; t; 0)$ is subabundant, then it is clear that, since the t simple points are assumed to be general, $S(\mathbf{n}; \mathbf{a}; s; 0; 0) = T(\mathbf{n}; \mathbf{a}; s)$ is true if and only if $S(\mathbf{n}; \mathbf{a}; s; t; 0)$ is true.

The following theorem describes the induction procedure we can apply to study Segre-Veronese varieties when one of the degree is one. This technique is inspired by the paper [3], where the authors study Segre varieties.

Theorem 2.13. *Let $a_1 = 1$, $n_1 = n_1' + n_1'' + 1$, $s = s' + s''$ and $t = t' + t''$, and let $\mathbf{n}' = (n_1', n_2, \ldots, n_k)$, $\mathbf{n}'' = (n_1'', n_2, \ldots, n_k) \in \mathbb{N}^k$. Suppose that $(\mathbf{n}'; \mathbf{a}; s'; t'; v + s'')$ and $(\mathbf{n}''; \mathbf{a}; s''; t''; v + s')$ are subabundant (resp. superabundant). If $S(\mathbf{n}'; \mathbf{a}; s'; t'; v + s'')$ and $S(\mathbf{n}''; \mathbf{a}; s''; t''; v + s')$ are true, then $(\mathbf{n}; \mathbf{a}; s; t; v)$ is subabundant (resp. superabundant) and $S(\mathbf{n}; \mathbf{a}; s; t; v)$ is true.*

Proof. We only focus on the case when $(\mathbf{n}'; \mathbf{a}; s'; t'; v + s'')$ and $(\mathbf{n}''; \mathbf{a}; s''; t''; v + s')$ are subabundant, because the remaining case can be proved in a similar fashion.

Let U be a $(n_1' + 1)$-dimensional subspace of V_1. Then we have the following Koszul complex:

$$\cdots \to (V_1/U)^* \otimes \mathcal{O}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{a}') \to \mathcal{O}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{a}) \to \mathcal{O}_{\mathbb{P}^{\mathbf{n}'}}(\mathbf{a}) \to 0.$$

By taking the cohomology, we obtain

$$0 \to (V_1/U)^* \otimes H^0(\mathcal{O}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{a}')) \to H^0(\mathcal{O}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{a})) \to H^0(\mathcal{O}_{\mathbb{P}^{\mathbf{n}'}}(\mathbf{a})) \to 0.$$

Taking the dual of the first linear transformation of the above sequence yields the rational map φ from $\prod_{i=1}^{k} \mathbb{P}^{n_i}$ to $\mathbb{P}^{\mathbf{n}''} = \mathbb{P}(V_1/U) \times \prod_{i=2}^{k} \mathbb{P}^{n_i}$.

Let $Z = \{p_1^2, \ldots, p_s^2\}$, let $\Phi = \{q_1, \ldots, q_t\}$ and let $\Psi = \{f_{r_1}, \ldots, f_{r_v}\}$. Suppose that $\{p_1, \ldots, p_{s''}\}$ and $\{q_1, \ldots, q_{t''}\}$ are not contained in $\mathbb{P}^{\mathbf{n}'}$, but the rest of the p_i's and q_i's are in $\mathbb{P}^{\mathbf{n}'}$, while the r_i's are general points. Then we have the following exact sequence:

$$0 \to \mathcal{I}_{Z\cup\Phi\cup\Psi\cup\mathbb{P}^{\mathbf{n}'}}(\mathbf{a}) \to \mathcal{I}_{Z\cup\Phi\cup\Psi}(\mathbf{a}) \to \mathcal{I}_{(Z\cup\Phi\cup\Psi)\cap\mathbb{P}^{\mathbf{n}'},\mathbb{P}^{\mathbf{n}'}}(\mathbf{a}) \to 0.$$

Let π'' be the canonical projection from $\mathbb{P}^{\mathbf{n}''}$ to $\prod_{i=2}^k \mathbb{P}^{n_i}$ and let Z'' be the following zero-dimensional subscheme of $\mathbb{P}^{\mathbf{n}"}$:

$$\{\varphi(p_1)^2, \ldots, \varphi(p_{s''})^2, \varphi(q_1), \ldots, \varphi(q_{t''}), f_{\varphi(r_1)}, \ldots, f_{\varphi(r_v)}, f_{\varphi(p_{s"+1})}, f_{\varphi(p_{s'})}\}.$$

One can show that $H^0(\mathbb{P}^{\mathbf{n}''}, \mathcal{I}_{Z''}(\mathbf{a}))$ is isomorphic to $H^0\left(\mathbb{P}^{\mathbf{n}}, \mathcal{I}_{Z\cup\Phi\cup\mathbb{P}^{\mathbf{n}'}}(\mathbf{a})\right)$.

Let ψ be the projection from $\mathbb{P}^{\mathbf{n}} \setminus \mathbb{P}^{\mathbf{n}''}$ to $\mathbb{P}^{\mathbf{n}'}$ and let Z' be the following zero-dimensional subscheme of $\mathbb{P}^{\mathbf{n}'}$:

$$\{p^2_{s''+1}, \ldots, p^2_s, q_{t''+1}, \ldots, q_t, f_{\psi(r_1)}, \ldots, f_{\psi(r_v)}, f_{\psi(p_1)}, \ldots, f_{\psi(p_{s''})}\}.$$

Note that $H^0(\mathcal{I}_{Z'}(\mathbf{a}))$ is isomorphic to $\left(I_{Z\cup\Phi} + I_{\mathbb{P}^{\mathbf{n}'}}/I_{\mathbb{P}^{\mathbf{n}'}}\right)_{\mathbf{a}}$. This implies that if

$$h_{\mathbb{P}^{\mathbf{n}''}}(Z'', \mathbf{a}) = s''\left(1 + n''_1 + \sum_{i=2}^k n_i\right) + t'' + (v + s')(n''_1 + 1)$$

and

$$h_{\mathbb{P}^{\mathbf{n}'}}(Z', \mathbf{a}) = s'\left(1 + n'_1 + \sum_{i=2}^k n_i\right) + t' + (v + s'')(n'_1 + 1)$$

then $h_{\mathbb{P}^{\mathbf{n}}}(Z \cup \Phi, \mathbf{a}) = s\left(1 + \sum_{i=1}^k n_i\right) + t + v(n_1+1)$, which completes the proof. $\square$

Example 2.14. As the first application of Theorem 2.13, we will show that $T(\mathbf{n}; \mathbf{a}; s)$ is true with $(\mathbf{n}; \mathbf{a}; s) = (2, 2; 1, 4; 9)$. Note that $(\mathbf{n}; \mathbf{a}; s)$ is subabundant. Let $s' = 6$. Then $s'' = 9 - 6 = 3$. Since $(0, 2; 1, 4; 3; 0; 6)$ and $(1, 2; 1, 4; 6; 0; 3)$ are equiabundant, we can reduce $T(\mathbf{n}; \mathbf{a}; s)$ to $S(0, 2; 1, 4; 3; 0; 6) = S(2; 4; 3; 0; 6)$ and $S(1, 2; 1, 4; 6; 0; 3)$. The statement $S(1, 2; 1, 4; 6; 0; 3)$ can be reduced to twice $S(0, 2; 1, 4; 3; 0; 6)$. In order to prove that $T(\mathbf{n}; \mathbf{a}; s)$ is true, it is therefore enough to prove the truth of $S(0, 2; 1, 4; 3; 0; 6)$. Note that $S(0, 2; 1, 4; 3; 0; 6)$ and $S(2; 4; 3; 6; 0)$ are the same statements. Also, the condition that $S(2; 4; 3; 6; 0)$ is true is equivalent to the condition that $S(2; 4; 3; 0; 0) = T(2; 4; 3)$ is true. It is known by the Alexander-Hirschowitz theorem that $T(2; 4; 3)$ is true. Thus $T(\mathbf{n}; \mathbf{a}; s)$ is also true.

Let $\mathbf{n}$, $\mathbf{a} \in \mathbb{N}^k$. As already stated in Section 1, Theorem 2.9 cannot be applied to any secant variety of $X_{\mathbf{n},\mathbf{a}}$ if $\mathbf{a} = (2^k)$. Theorem 2.13 cannot be used directly in this case either. In the following example, we illustrate how to combine an argument based on the Castelnuovo inequality with Theorem 2.13 to study secant varieties of such Segre-Veronese varieties:

Example 2.15. Here we prove that $T(2, 2; 2, 2; 5)$ is true. Let $p_1, \ldots, p_5$ be generic points of $(\mathbb{P}^2)^2$ and let $Z = \{p_1^2, \ldots, p_5^2\}$. Specializing p_1, p_2 and p_3 to $H = \mathbb{P}^1 \times \mathbb{P}^2 \subset (\mathbb{P}^2)^2$ yields a short exact sequence

$$0 \to \mathcal{I}_{\widetilde{Z}}(1, 2) \to \mathcal{I}_Z(2, 2) \to \mathcal{I}_{Z\cap H,H}(2, 2) \to 0.$$

It was shown by Bauer and Draisma [8] that $h_H(Z \cap H, (2, 2))$ has the expected value, i.e., $T(1, 2; 2, 2; 3)$ is true. It suffices therefore to show that $\widetilde{Z}$ has the expected value at $(1, 2)$. Note that $\widetilde{Z} = \{p_1, p_2, p_3, p_4^2, p_5^2\}$. Recall that p_1, p_2 and p_3 lie in H. Thus specializing p_5 to H, we can reduce the above-mentioned statement to $S(1, 2; 1, 2; 1; 3; 1)$ and $S(0, 2; 1, 2; 1; 0; 1)$. Note that $S(0, 2; 1, 2; 1; 0; 1)$ is equivalent to $S(2;2;1;1;1;0)$. Since $S(2;2;1;1;0;0)$ is true, so is $S(2;2;1;1;1;0)$ by Remark 2.12. Thus it remains to show that $S(1,2;1,2;1;3;1)$ is true. This statement can be reduced to $S(0,2;1,2;1;1;1)$ and $S(0,2;1,2;0;2;2)$. By Remark 2.12, $S(0,2;1,2;1;1;1)$ and $S(0,2;1,2;0;2;2)$ are equivalent to $S(2;2;1;2;0)$ and $S(2;2;0;4;0)$ respectively. Clearly, the latter statement is true. Also, since $S(2;2;1;0;0)$ is true, so is $S(2;2;1;2;0)$. Thus $S(1,2;1,2;1;3;1)$ is true. Therefore, $T(1,2;2,2;3)$ is true.

We conclude this section by presenting immediate, but useful consequences of Theorem 2.4 and Lemma 2.5.

Lemma 2.16. *Let* $\mathbf{a},\mathbf{b},\mathbf{n},\mathbf{m} \in (\mathbb{Z}_{\geq 0})^k \setminus \{(0,\dots,0)\}$ *and let* $s \in \mathbb{N}$. *Suppose that* $\mathbf{a} \leq \mathbf{b}$ *and* $\mathbf{n} \leq \mathbf{m}$.

(i) *If* $T(\mathbf{n};\mathbf{a};s)$ *is true, if* $(\mathbf{n};\mathbf{a};s)$ *is subabundant and if* $\mathbf{a} \geq (1,\dots,1)$, *then* $T(\mathbf{n};\mathbf{b};s)$ *is true and* $(\mathbf{n};\mathbf{b};s)$ *is subabundant.*

(ii) *If* $T(\mathbf{n};\mathbf{a};s)$ *is true, if* $(\mathbf{n};\mathbf{a};s)$ *is subabundant and if*

$$s \leq \binom{n_\ell + a_\ell - 1}{a_\ell - 1} \prod_{i \neq \ell} \binom{n_i + a_i}{a_i}$$

for any ℓ *such that* $m_\ell > n_\ell$ *and* $a_\ell \geq 1$, *then* $T(\mathbf{m};\mathbf{a};s)$ *is true and* $(\mathbf{m};\mathbf{a};s)$ *is subabundant.*

(iii) *If* $T(\mathbf{n};\mathbf{b};s)$ *is true and if* $(\mathbf{n};\mathbf{b};s)$ *is superabundant, then* $T(\mathbf{n};\mathbf{a};s)$ *is true and* $(\mathbf{n};\mathbf{a};s)$ *is superabundant.*

Proof. Note that if $(\mathbf{n};\mathbf{a};s)$ is subabundant, then so are $(\mathbf{n};\mathbf{b};s)$ and $(\mathbf{m};\mathbf{a};s)$.

(i) Since $\mathbf{a} \geq (1,\dots,1)$, without loss of generality, we may assume that $1 \leq a_1 < b_1$. Let $\mathbf{a}' = (a_1+1, a_2, \dots, a_k)$ and H a hyperplane defined by a linear form in $R_{(1,0,\dots,0)}$. By induction it suffices to prove that $T(\mathbf{n};\mathbf{a}';s)$ is true. Consider a collection Z of s general double points in $\mathbb{P}^{\mathbf{n}}$. Suppose that the support of Z is not contained in H. From the short exact sequence,

$$0 \to \mathcal{I}_Z(\mathbf{a}) \to \mathcal{I}_Z(\mathbf{a}') \to \mathcal{O}_H(\mathbf{a}') \to 0,$$

we can conclude that $h_{\mathbb{P}^{\mathbf{n}}}(Z,\mathbf{a}')$ is the expected value, because the trace of Z is empty and $\widetilde{Z} = Z$.

(ii) The statement is trivial if $\mathbf{n} = \mathbf{m}$. Thus we may assume that $\mathbf{n} < \mathbf{m}$. Then there exists at least one $\ell \in \{1,\dots,k\}$ such that $m_\ell > n_\ell$ and $a_\ell \geq 1$, because otherwise $T(\mathbf{n};\mathbf{a};s)$ and $T(\mathbf{m};\mathbf{a};s)$ are the same statement. Without loss of generality, we may assume that $\ell = 1$. Then, by induction, it is enough to prove that $T(\mathbf{n}';\mathbf{a};s)$ is true for $\mathbf{n}' = (n_1+1, n_2, \dots, n_k)$. Consider a collection Z of s general double points of $\mathbb{P}^{\mathbf{n}'}$. Suppose that they are all contained in $H = \mathbb{P}^{\mathbf{n}} \subset \mathbb{P}^{\mathbf{n}'}$. Hence the trace of Z is given by s double points of H, while the residual $\widetilde{Z}$ is given by s simple points contained in the hyperplane H. Then we have the following exact sequence

$$0 \to \mathcal{I}_{\widetilde{Z}}(a_1 - 1, a_2, \dots, a_k) \to \mathcal{I}_Z(\mathbf{a}) \to \mathcal{I}_{Z \cap H}(\mathbf{a}) \to 0,$$

By assumption, $T(\mathbf{n};\mathbf{a};s)$ is true. Thus (ii) immediately follows from Lemma 2.5 and from the assumption that $s \leq \binom{n_1+a_1-1}{a_1-1} \prod_{i=2}^{k} \binom{n_i+a_i}{a_i}$.

(iii) Clearly if $(\mathbf{n};\mathbf{b};s)$ is superabundant, then $(\mathbf{n};\mathbf{a};s)$ is also superabundant. Given $\mathbf{b}' = (b_1 - 1, b_2, \dots, b_h)$, we only need to prove $T(\mathbf{n};\mathbf{b}';s)$. As in the proof of (i), we consider a collection Z of s general double points whose support is not contained in H. Then, by the Castelnuovo exact sequence, we can immediately see that $T(\mathbf{n};\mathbf{b}';s)$ is true. $\square$

3. Two-factor Segre-Veronese varieties

The purpose of this section is to establish the existence of a class of non-defective two-factor Segre-Veronese varieties. First of all, we will recall some basic results on secant varieties of such Segre-Veronese varieties. Let $\mathbf{n}=(m,n)$, $\mathbf{a}=(a,b)\in\mathbb{N}^2$ and let $N(\mathbf{n},\mathbf{a})=\binom{m+a}{a}\binom{n+b}{b}$. We use just N instead of $N(\mathbf{n};\mathbf{a})$ if $\mathbf{n}$ and $\mathbf{a}$ are clear from the context. As in the previous section, we denote by $X_{\mathbf{n},\mathbf{a}}$ the Segre-Veronese variety obtained from $\mathbb{P}^{\mathbf{n}}$ by embedding in $\mathbb{P}^{N-1}$ by the morphism given by $\mathcal{O}(\mathbf{a})$. Let $\underline{s}(\mathbf{n},\mathbf{a})=\lfloor N/(m+n+1)\rfloor$ and let $\overline{s}(\mathbf{n},\mathbf{a})=\lceil N/(m+n+1)\rceil$. We write $\underline{s}$ and $\overline{s}$ instead of $\underline{s}(\mathbf{n},\mathbf{a})$ and $\overline{s}(\mathbf{n},\mathbf{a})$ respectively if $\mathbf{n}$ and $\mathbf{a}$ are clear from the context. As mentioned in Remark 2.12, in order to prove that $T(\mathbf{n};\mathbf{a})$ is true, it is sufficient to show that $T(\mathbf{n};\mathbf{a};s)$ for $s=\underline{s}$ and $\overline{s}$.

As was mentioned earlier, the problem of finding the dimension of $\sigma_s(X_{\mathbf{n},\mathbf{a}})$ can be translated into the problem of calculating the value of the multi-graded Hilbert function of s double points on $\mathbb{P}^{\mathbf{n}}$ at $\mathbf{a}$. In their several papers, Catalisano, Geramita, and Gimigliano showed the relationship between ideals of varieties in multi-projective space and ideals in standard polynomial rings. In [16, Theorem 2.1], they used it to prove the following theorem:

Theorem 3.1 ([16])**.** *$T(1,1;a,b;s)$ is true except for $a=2$, $b=2d$ ($d\geq 1$) and $s=b+1$.*

This theorem was also proved by Baur and Draisma. Their proof uses tropical techniques (see [8, Theorem 1.1] for more details).

Example 3.2. As the first application of our techniques we prove that $T(m,1;1,2;2)$ is true for any $m\geq 1$.

By Theorem 3.1, $T(1,1;1,2;2)$ is true. Moreover $(1,1;1,2;2)$ is equiabundant. Since

$$s=2<3=\binom{n_1+a_1-1}{a_1-1}\binom{n_2+a_2}{a_2},$$

we can deduce that $T(m,1;1,2;2)$ are true for all $m\geq 1$ by Lemma 2.16 (ii).

Let s be a positive integer and let s' and ε be the quotient and remainder when dividing $s(m+n+1)-\binom{m+a-1}{a-1}\binom{n+b}{b}$ by $m+n$. In order to prove the truth of $T(m,n;a,b;s)$, we need to show that the 5-tuple $(m,n;a-2,b;s-s'-\varepsilon)$ is superabundant. The following lemma proves that this is actually the case for most of $(m,n;a,b)$:

Lemma 3.3. *Let $a,b\geq 3$. For each $1\leq s\leq\left\lceil\frac{\binom{m+a}{a}\binom{n+b}{n}}{m+n+1}\right\rceil$, let s' and ε be as above. Then $(m,n;a-2,b;s-s'-\varepsilon)$ is superabundant unless $(m,n)=(1,1)$.*

Proof. We want to prove that the integer $F(m,n,a,b)$ is non-negative, where

$$F(m,n;a,b)=(s-s'-\varepsilon)(m+n+1)-N(m,n;a-2,b). \tag{6}$$

By definition,

$$s\,(m+n+1)-N(m,n;a-1,b)=s'\,(m+n)+\varepsilon, \tag{7}$$

where $0 \le \varepsilon \le m+n-1$. So we have

$$\begin{aligned} & F(m,n;a,b) \\ = \;& s\,(m+n+1) - s'\,(m+n) - \varepsilon - s' - \varepsilon\,(m+n) - N(m,n;a-2,b) \\ = \;& N(m,n;a-1,b) - N(m,n;a-2,b) - s' - \varepsilon\,(m+n) \\ = \;& N(m-1,n;a-1,b) - s' - \varepsilon\,(m+n) \end{aligned}$$

Since $s\,(m+n+1) \le N(m,n;a,b) + (m+n+1)$ by assumption, the following inequality holds:

$$s\,(m+n+1) - N(m,n;a-1,b) \le N(m-1,n;a,b) + (m+n+1).$$

This implies that $s'\,(m+n) \le N(m-1,n;a,b) + (m+n+1)$, i.e.,

$$s' \le \frac{1}{m+n}\left\{N(m-1,n;a,b) + (m+n+1)\right\}.$$

Thus we obtain

$$\begin{aligned} & F(m,n;a,b) \\ \ge \;& N(m-1,n;a-1,b) - \frac{\{N(m-1,n;a,b)+(m+n+1)\}}{m+n} - \varepsilon\,(m+n) \\ \ge \;& \frac{\binom{n+b}{n}}{m+n} H(m,n;a,b) - \frac{m+n+1}{m+n} - (m+n-1)(m+n), \end{aligned}$$

where $H(m,n;a,b) = \frac{(m+a-2)!}{a!(m-1)!}\{a(m+n) - (m+a-1)\}$. Note that $H(m,n;a,b)$ is an increasing function of a if $a \ge 3$. So we have $F(m,n;a,b) \ge F(m,n,3,3)$. Let

$$G(m,n;a,b) = \frac{\binom{n+b}{n}}{m+n} H(m,n;a,b) - \frac{m+n+1}{m+n} - (m+n-1)(m+n).$$

It is not very hard to show that $G(m,n;3,3) \ge -\frac{2}{3}$ unless $(m,n) = (1,1)$. Thus $F(m,n;a,b) \ge F(m,n;3,3) \ge 0$. $\square$

In the following lemma, we show that the inequality $s' \ge \varepsilon$ holds in most cases:

Lemma 3.4. *Let $a \ge 3$ and let $b,m,n \ge 1$. For each $s \ge \left\lfloor \frac{\binom{m+a}{a}\binom{n+b}{n}}{m+n+1} \right\rfloor$, let s' and ε be as above. Then $s' \ge \varepsilon$ in the following cases:*

(i) *$b \ge 3$;*
(ii) *$b = 1$ and $m \ge 3$;*
(iii) *$b = 1$, $m = 2$ and $n = 1$.*

Proof. Since $n+m-1 \ge \varepsilon$, it suffices to show that $s' \ge n+m-1$. Assume that $s' < n+m-1$. By assumption we know that $s(m+n+1) \ge \binom{m+a}{a}\binom{n+b}{n} - (m+n)$. Combining this relation with (7) yields

$$N(m,n;a,b) - (m+n) \le s'\,(m+n) + \varepsilon + N(m,n;a-1,b)$$

from which we obtain

$$\begin{aligned} N(m-1,n;a,b) \;&\le\; s'\,(m+n) + \varepsilon + (m+n) \\ &\le\; (n+m-2)(m+n) + (m+n-1) + (m+n) \\ &=\; (n+m)^2 - 1. \end{aligned}$$

Now we need to prove that this inequality provides a contradiction in each case. Let $G(m,n,a,b) = N(m-1,n;a,b)-(m+n)^2$. It is enough to prove that $G(m,n,a,b) > 0$.

(i) Suppose that $b \geq 3$. Note that $N(m-1,n;a,b) \geq N(m-1,n;3,3)$. It follows therefore that if $a,b \geq 3$, then $G(m,n,a,b) \geq G(m,n;3,3)$. It is straightforward to prove that $G(m,n,3,3)$ is positive for all $m,n \geq 1$.

(ii) Suppose that $b = 1$ and $m \geq 3$. In the same way as in (i), one can prove that $G(m,n,a,b) \geq G(m,n;3,1)$. It is not hard to show that $G(m,n,3,1)$ is positive when $m \geq 3$, and $n \geq 1$.

(iii) Assume that $b = 1, m = 2$ and $n = 1$. Then we have $\varepsilon \leq 2$. We want to prove that $s' \geq 2$. Assume that $s' \leq 1$. By the hypothesis we have $s \geq \left\lfloor \frac{(a+2)(a+1)}{4} \right\rfloor$, which implies $4s \geq (a+2)(a+1)-3$. By (7) we have

$$(a+2)(a+1)-3 \leq a(a+1)+3s'+\varepsilon \leq a(a+1)+3+2,$$

or $2(a-3) \leq 0$, which is false for all $a \geq 4$. If $a = 3$, then we have $s \geq 5$. On the other hand, (7) gives rise to $4s \leq 12+3+2 = 17$, which is a contradiction. $\square$

The result presented below was already proved by Chiantini and Ciliberto [20]. Here we give a different proof to illustrate how the Horace method works.

Theorem 3.5. *$T(n,1;1,d)$ is true for any $n,d \geq 1$.*

Proof. The proof is by induction on d. It is immediate to check that $T(n,1;1,1)$ is true (see Section 1). The truth of the statement $T(n,1;1,2)$ immediately follows from Example 3.2 and [2, Example 2.9]. Thus we may assume $d \geq 3$ and $n \geq 1$.

We first prove the truth of $T(n,1;1,d;s)$ for $s = \underline{s}(n,1;1,d) = \left\lfloor \frac{(n+1)(d+1)}{n+2} \right\rfloor$. Let $p_1,\ldots,p_s$ be points on $\mathbb{P}^n \times \mathbb{P}^1$ and let $Z = \{p_1^2,\ldots,p_s^2\}$. Suppose that p_s lies in a hyperplane H of degree $(0,1)$. Then we get the following sequence:

$$0 \to \mathcal{I}_{\widetilde{Z}}(1,d-1) \to \mathcal{I}_Z(1,d) \to \mathcal{I}_{Z\cap H,H}(1,d) \to 0,$$

where $\widetilde{Z} = \{p_1^2,\ldots,p_{s-1}^2\} \cup \{p_s\}$. Since the trace of Z consists of only one double point of H, we have

$$h_H(Z\cap H,(1,d)) = n+1.$$

By induction hypothesis, $T(n,1;1,d-1;s-1)$ and $T(n,1;1,d-2;s-1)$ are both true, and thus

$$h_{\mathbb{P}^{\mathbf{n}}}(\{p_1^2,\ldots,{p_{s-1}}^2\},(1,d-1)) = \min\{(s-1)(n+2),(n+1)d\} = (s-1)(n+2)$$

and

$$h_{\mathbb{P}^{\mathbf{n}}}(\{p_1^2,\ldots,{p_{s-1}}^2\},(1,d-2)) = \min\{(s-1)(n+2),(n+1)(d-1)\}.$$

It is straightforward to prove the inequality

$$(s-1)(n+2)+1 \leq \min\{(s-1)(n+2),(n+1)(d-1)\}+(n+1).$$

So it follows from Lemma 2.5 that $h_{\mathbb{P}^{\mathbf{n}}}(\widetilde{Z},(1,d-1)) = (s-1)(n+2)+1$. By Theorem 2.4 we can deduce that $h_{\mathbb{P}^{\mathbf{n}}}(Z,(1,d)) = s(n+2)$, because $h_{\mathbb{P}^{\mathbf{n}}}(\widetilde{Z},\mathbf{a}')$ and $h_{\mathbb{P}^{\mathbf{n}'}}(Z\cap H,\mathbf{a})$ are the expected values and they are both subabundant. Thus, $T(n,1;1,d;s)$ is true.

In a similar manner, we can prove that $T(n,1;1,d;s)$ is true for $s=\overline{s}(n,1;1,d)$. Let $p_1,\dots,p_s$ be points on $\mathbb{P}^n\times\mathbb{P}^1$ and let $Z=\{p_1^2,\dots,p_s^2\}$. Specializing p_s to H yields the following sequence:

$$0\to\mathcal{I}_{\widetilde{Z}}(1,d-1)\to\mathcal{I}_Z(1,d)\to\mathcal{I}_{Z\cap H,H}(1,d)\to 0.$$

As in the previous case, we have $h_H(Z\cap H,(1,d))=n+1$. By induction hypothesis, $T(n,1;1,d-1;s-1)$ is true. Additionally, $(n,1;1,d-1;s-1)$ is superabundant. Therefore, $h_{\mathbb{P}^n}(Z,(1,d))=(n+1)(d+1)$, which completes the proof. $\square$

We recall now a result proved by Abrescia.

Theorem 3.6 ([5])**.** *$T(n,1;2,2d+1)$ is true for any $n\geq 1$ and $d\geq 0$.*

The following is the first application of the differential Horace lemma:

Theorem 3.7. *$T(n,1;a,2d+1)$ is true for any $d,n,a\geq 1$.*

Proof. The proof is by double induction on n and a. We know that $T(n,1;1,2d+1)$ is true by Theorem 3.5 and that $T(n,1;2;2d+1)$ is true by Theorem 3.6. The statement $T(1,1;a,2d+1)$ is also true by Theorem 3.1.

Suppose now that $a\geq 3$ and $n\geq 2$. Recall that it is enough to prove $T(n,1;a,2d+1;s)$ for $s\in\{\underline{s},\overline{s}\}$. We want to apply Theorem 2.9. Let s' and ϵ be the quotient and remainder when dividing $s(n+2)-N(n,1;a-1,2d+1)$ by $n+1$. Note that, by Lemma 3.3, $(n,1;a-2,2d+1;s-s'+\varepsilon)$ is superabundant, because $n\geq 2$, $a\geq 3$ and $2d+1\geq 3$. Additionally, by Lemma 3.4 (i), we obtain $s'\geq\varepsilon$. Now, by induction hypothesis, $T(n-1,1;a,2d+1)$, $T(n,1;a-1,2d+1)$ and $T(n,1;a-2,2d+1)$ are all true. Thus Theorem 2.9 implies that $T(n,1;a,2d+1)$ is true. $\square$

The following theorem is a consequence of Theorem 2.13:

Theorem 3.8. *For any $n,d\geq 1$, $T(n,1;2,2d;s)$ is true if $s\leq d(n+1)$ or $s\geq (d+1)(n+1)$.*

Proof. To prove this theorem, we only need to show that $T(n,1;2,2d;d(n+1))$ and $T(n,1;2,2d;(d+1)(n+1))$ are true. The proof is by induction on n. Recall that $T(1,1;2,2d)$ is true unless $s=2d+1$ by Theorem 3.1. Also, $T(n,1;1,2d)$ is true by Theorem 3.5.

We first prove that $T(n,1;2,2d;d(n+1))$ is true. Let $s=dn+d$, $s'=dn$, $s''=d$ and let H be a hyperplane of multi-degree $(1,0)$.

Specializing s' points to H, since $(n-1,1;2,2d;s')$ and $(n,1;1,2d;s'';s';0)$ are both subabundant, we can apply Theorem 2.4. By induction hypothesis, $T(n-1,1;2,2d;s')$ is true and so it suffices to prove that $h_{\mathbb{P}^n\times\mathbb{P}^1}(\widetilde{Z},(1,2d))$ is the expected value, where $\widetilde{Z}$ is given by s'' general double points and s' simple points contained in H (and general). In order to prove this fact we apply now Theorem 2.13. Since $(n-1,1;1,2d;0;s';s'')$ and $(0,1;1,2d;s'';0;0)$ are both subabundant, it is enough to prove that $S(n-1,1;1,2d;0;s';s'')$ and $S(0,1;1,2d;s'';0;0)$ are true.

By Theorem 3.5, $S(n-1,1;1,2d;s'';0;0)=T(n-1,1;1,2d;s'')$ is true. This implies that $S(n-1,1;1,2d;0;0;s'')$ is also true. Additionally, the s' points are in general position in H. So $S(n-1,1;1,2d;0;s';s'')$ is true. Since $S(0,1;1,2d;d;0;0)=T(1;2d;d)$ is clearly true, the theorem follows from Theorem 2.13.

One can prove that $T(n,1;2,2d;(d+1)(n+1))$ is true by taking $s'=(d+1)n$ and $s''=d+1$ and by replacing "subabundant" by "superabundant" in the previous argument. $\square$

Remark 3.9. In [5] Abrescia proved Theorem 3.8 with different techniques. Moreover she proved that $\sigma_s(X_{(n,1),(2,2d)})$ is defective for any $d(n+1)+1 \leq s \leq (d+1)(n+1)-1$.

Lemma 3.10. *$T(n,1;3,4)$ is true for any $n \geq 1$.*

Proof. To prove this lemma, it is enough to show that $T(n,1;3,4;s)$ is true for $s \in \{\underline{s}, \overline{s}\}$. Here we only show that $T(n,1;3,4;s)$ is true for $s = \underline{s}$, because the remaining case follows the same path. The proof is by induction on n. Note that $T(1,1;3,4)$ is true by Theorem 3.1.

Suppose now that $n \geq 2$. We also assume by induction that $T(n-1,1;3,4)$ is true. Let s' and ε be the quotient and remainder in the division of $s(n+2) - 5\binom{n+2}{2}$ by $n+1$. Then, in order to apply Theorem 2.9, it is enough to check that $T(n,1;2,4;s-s')$ and $T(n,1;1,4;s-s'-\varepsilon)$ are true, that $(n,1;1,4;s-s'-\varepsilon)$ is superabundant and that $s' \leq \varepsilon$.

From Theorem 3.5 it follows that $T(n,1;1,4)$ is true. Moreover $(n,1;1,4;s-s'-\varepsilon)$ is superabundant by Lemma 3.3, because $n \geq 2$, and $s' \geq \varepsilon$ by Lemma 3.4 (i). By Theorem 3.8, $T(n,1;2,4;s-s')$ is true if $s-s' \leq 2(n+1)$. Hence our task is to show that the inequality $s' \leq \varepsilon$ holds.

It is not hard to prove that the inequality holds for $n=2$, and so we may assume that $n \geq 3$. By the definitions of s and s', we have

$$\begin{aligned}
2n+2-s+s' &= 2n+2-\left\lfloor \frac{5\binom{n+3}{3}}{n+2} \right\rfloor + \left\lfloor \frac{\left\lfloor \frac{5\binom{n+3}{3}}{n+2} \right\rfloor (n+2) - 5\binom{n+2}{2}}{n+1} \right\rfloor \\
&= 2n+2+\left\lfloor \frac{\left\lfloor \frac{5(n+3)(n+1)}{6} \right\rfloor - 5\binom{n+2}{2}}{n+1} \right\rfloor \\
&> 2n+2+\frac{\left\lfloor \frac{5(n+3)(n+1)}{6} \right\rfloor - 5\binom{n+2}{2}}{n+1} - 1 \\
&\geq 2n+1+\frac{5(n+3)}{6} - \frac{1}{n+1} - \frac{5(n+2)}{2} \\
&= \frac{2n^2-7n-15}{6(n+1)}.
\end{aligned}$$

It is straightforward to show that $f(n) = \frac{2n^2-7n-15}{6(n+1)}$ is an increasing function. Since $f(3) = -3/4$, we can conclude that $2n+2-s+s' \geq 0$. Thus we completed the proof. $\square$

Lemma 3.11. *$T(n,1;4,4)$ is true for any $n \geq 1$.*

Proof. In order to prove the truth of this statement, it is enough to show that $T(n,1;4,4;\underline{s})$ and $T(n,1;4,4;\overline{s})$ are true. Here we only consider the first case, because the remaining case can be proved in a similar fashion.

We use induction on n. Note that $T(1,1;4,4;s)$ is true by Theorem 3.1. It can be also proved directly that $T(2,1;4,4;s)$ is true. So we may assume that $n \geq 3$. Let s' and ε be the quotient and remainder in the division of $s(n+2) - 5\binom{n+3}{3}$ by $n+1$ respectively. Since $(n,1;2,4;s-s'-\varepsilon)$ is superabundant by Lemma 3.3 and $s' \geq \varepsilon$ by Lemma 3.4 (i), the statement $T(n,1;4,4;s)$ can be reduced to $T(n-1,1;4,4;s')$, $T(n,1;3,4;s-s')$ and $T(n,1;2,4;s-s'-\varepsilon)$. By induction hypothesis,

$T(n-1,1;4,4;s')$ is true. It follows from Lemma 3.10 that $T(n,1;3,4;s-s')$ is true. Hence it suffices to prove that the inequality $s-s'-\varepsilon \geq 3n+3$ holds by Theorem 3.8.

It is not hard to show that the above inequality holds for $n=2$. Suppose therefore that $n \geq 3$. Then

$$\begin{aligned}
s-s'-\varepsilon &= \left\lfloor \frac{5\binom{n+4}{4}}{n+2} \right\rfloor - \left\lfloor \frac{\left\lfloor \frac{5\binom{n+4}{4}}{n+2} \right\rfloor (n+2) - 5\binom{n+3}{3}}{n+1} \right\rfloor - \epsilon \\
&= -\left\lfloor \frac{\left\lfloor \frac{5\binom{n+4}{4}}{n+2} \right\rfloor - 5\binom{n+3}{3}}{n+1} \right\rfloor - \epsilon \\
&\geq \frac{-\left\lfloor \frac{5\binom{n+4}{4}}{n+2} \right\rfloor + 5\binom{n+3}{3}}{n+1} - n \\
&\geq -\frac{5(n+4)(n+3)}{24} + \frac{5(n+3)(n+2)}{6} - n \\
&= \frac{15n^2+41n+60}{24}.
\end{aligned}$$

One can readily show that $\frac{15n^2+41n+60}{24} \geq 3n+3$ if $n \geq 3$. Thus we completed the proof. □

Theorem 3.12. *$T(n,1;a,4)$ is true for any $n \geq 1$ and $a \geq 3$.*

Proof. The proof is by induction on n and a. Note that, since $a \geq 3$, $T(1,1;a,4)$ is true by Theorem 3.1. We have also proved that $T(n,1;3,4)$ and $T(n,1;4;4)$ are true for any $n \geq 1$ (see Lemmas 3.10 and 3.11).

Assume now that $n \geq 2$ and $s \in \{\underline{s}, \overline{s}\}$. Let s' and ε be the quotient and remainder in the division of $s(n+1) - 5\binom{n+a}{a}$ by $n+1$ respectively. Note that $(n,1;a-2,4;s-s'-\varepsilon)$ is superabundant by Lemma 3.3, and $s' \geq \varepsilon$ by Lemma 3.4 (i). Thus $T(n,1;a,4;s)$ can be reduced to $T(n-1,1;a,4;s')$, $T(n,1;a-1,4;s-s')$ and $T(n,1;a-2,4;s-s'-\varepsilon)$. By induction hypotheses, these statements are all true. The statement $T(n,1;a,4;s)$ is therefore true by Theorem 2.9. □

Theorem 3.13. *If $a,b \geq 3$, then $T(n,1;a,b)$ is true for any $n \geq 1$.*

Proof. The statement $T(1,1;a,b)$ is true by Theorem 3.1, because $a,b \geq 3$. Suppose now that $n \geq 2$. The proof is by induction on b. Note that $T(n,1;a,3)$ is true by Theorem 3.7 and $T(n,1;a,4)$ is true by Theorem 3.12 for any $a \geq 3$. Thus we may assume that $b \geq 5$. It is enough to prove $T(n,1;a,b;\underline{s})$ and $T(n,1;a,b;\overline{s})$. Assume that $s \in \{\underline{s}, \overline{s}\}$. Let s' and ε be the quotient and remainder in the division of $s(n+2) - \binom{n+a}{a}b$ by $n+1$. By induction hypothesis, $T(n,1;a,b-1;s-s')$ and $T(n,1;a,b-2;s-s'-\varepsilon)$ are true. Additionally, Lemma 3.3 implies that $(n,1;a,b-2;s-s'-\varepsilon)$ is superabundant, because $n \geq 2$. Lemma 3.4 (i) implies that $s' \geq \varepsilon$, since $a,b \geq 3$. Note that $T(n,0;a,b;s')$ is true if $n \geq 2$ with only four exceptions by the Alexander-Hirschowitz theorem. Thus the statement follows immediately from Theorem 2.9 if $(n,a,s') \notin \{(2,4,5),(3,4,9),(4,3,7),(4,4,14)\}$.

Since s' and ε are the quotient and remainder in the division by $n+1$ respectively, we have the following equality:

$$(n+1)s' + \varepsilon = s(n+2) - \binom{n+a}{a} b \quad \text{for } 0 \le \varepsilon \le n. \tag{8}$$

If $n = 3$ and $a = 4$, then $s = \underline{s} = \overline{s} = 7(b+1)$, and thus $s' = 8$. So the above argument implies that $T(3,1;4,b;s)$, and hence $T(3,1;4,b)$, is true for every b. The same idea, however, cannot be applied to $(n,a) = (2,5)$, $(4,3)$ and $(4,4)$. Therefore, for each $(n,a) \in \{(2,4),(4,3),(4,4)\}$ and for each s such that

$$\left(n, a, \frac{s(n+2) - \binom{n+a}{a} b - \varepsilon}{n+1}\right)$$

falls into one of the above cases, we need to prove that $T(n,1;a,b;s)$ holds in a different way.

Let t and δ be the quotient and remainder in the division of $s(n+2) - \binom{n+a-1}{a-1}(b+1)$ by $n+1$ respectively. Note that $(n,1;a-2,b;s-t-\delta)$ is superabundant by Lemma 3.3 and $t \ge \delta$ by Lemma 3.4 (i). So in order to apply Theorem 2.9, we need only to check that $T(n-1,1;a,b;t)$, $T(n,1;a-1,b;s-t)$ and $T(n,1;a-2,b;s-t-\delta)$ are true. Below we will consider the above-mentioned three cases separately.

(i) Let $(n,a,s') = (2,4,5)$. From (8), we have $15 + \varepsilon = 4s - 15b$ and $0 \le \varepsilon \le 2$. This implies that we can assume that s is an integer of the form $s = \frac{15(b+1)+\varepsilon}{4}$ for some $\varepsilon \in \{0,1,2\}$. It suffices to prove the truth of $T(2,1;4,b;s)$ for such an s.

Let t and δ be the quotient and remainder in the division of $4s - 10(b+1)$ by 3 respectively. Note that $T(1,1;4,b;t)$ is true by Theorem 3.1, $T(2,1;3,b;s-t)$ is true for the first part of the proof. Moreover $T(2,1;2,b;s-t-\delta)$ holds by Lemma 3.6 if b is odd. By Theorem 3.8, the statement is also true for $b = 2k$ if $s - t - \delta \ge 3(k+1)$. Therefore, we need only to verify that this inequality holds. Assume that $b = 2k$. Then $\varepsilon = 1$ and $15b \equiv 0 \pmod 4$, which implies that k is even. Set $k = 2\ell$, so that $b = 4\ell$ and $s = 15\ell + 4$. Note that we may assume that $\ell \ge 2$, because $b \ge 5$. Thus $t = \lfloor \frac{4s-10(b+1)}{3} \rfloor = \lfloor \frac{20\ell+6}{3} \rfloor \le 7\ell + 2$. By definition, $\delta \le 2$. Thus we have $s - t - \delta \ge (15\ell+4) - (7\ell+2) - 2 = 8\ell \ge 6\ell + 3 = 3(k+1)$.

(ii) Let $(n,a,s') = (4,3,7)$. From (8), we have $35 + \varepsilon = 6s - 35b$ and $0 \le \varepsilon \le 4$. This implies that we may assume that s is an integer of the form $s = \frac{35(b+1)+\varepsilon}{6}$ with $0 \le \varepsilon \le 4$. Thus we need to prove $T(4,1;3,b;s)$ for such an s.

Let t and δ be the quotient and remainder in the division of $6s - 15(b+1)$ by 5. We have that $T(3,1;3,b;t)$ holds by the first part of the proof. by Lemma 3.6 $T(4,1;2,b;s-t)$ is true if b is odd. Additionally, if $b = 2k$ is even, $T(4,1;2,b;s-t)$ holds by Theorem 3.8 if $s - t \le 5k$. Thus it remains only to prove that this inequality holds. Since we know that $\varepsilon \le 4$, we need to take the following two cases into account:

(a) $\varepsilon = 1$, $k = 3h$, $b = 6h$, $s = 35h + 6$;
(b) $\varepsilon = 3$, $k = 3h+1$, $b = 6h+2$, $s = 35h + 18$.

In case (a), $t = \lfloor \frac{6s-15(b+1)}{5} \rfloor = 24h + \lfloor \frac{21}{5} \rfloor \ge 24h + 4$. Thus $s - t \le (35h+6) - (24h+4) = 11h + 2 \le 15h = 5k$. In case (b), $t = \lfloor \frac{6s-15(b+1)}{5} \rfloor \ge 24h + 12$, and hence $s - t \le (35h+18) - (24h+12) = 11h + 6 \le 5(3h+1) = 5k$.

(iii) Let $(n, a, s') = (4, 4, 14)$. From (8), we have $70 + \varepsilon = 6s - 70b$ and $0 \leq \varepsilon \leq 4$. This implies that $\varepsilon = 2\varepsilon'$ is even, and we may assume that s is an integer of the form $s = \frac{35(b+1)+\varepsilon'}{3}$ for $\varepsilon' = 0, 1, 2$.

Let t and δ be the quotient and remainder in the division of $6s - 35(b + 1)$ by 5. We have already shown that $T(3, 1; 4, b; t)$ and $T(4, 1; 3, b; s - t)$ are true. Thus we only need to prove $T(4, 1; 2, b; s - t - \delta)$. By Lemma 3.6 and Theorem 3.8 , this statement holds either if b is odd or if $b = 2k$ and $s - t - \delta \geq 5(k + 1)$. Thus all we have to do is prove the above inequality holds. To do so, we consider only the following three cases: are possible:

(a) $k = 3\ell$, $\varepsilon' = 1$, $b = 6\ell$, $s = 70\ell + 12$,
(b) $k = 3\ell + 1$, $\varepsilon' = 0$, $b = 6\ell + 2$, $s = 70\ell + 35$,
(c) $k = 3\ell + 2$, $\varepsilon' = 2$, $b = 6\ell + 4$, $s = 70\ell + 59$.

In all these three cases it is straightforward to show that $s - t - \delta \geq 5(k + 1)$ holds. Thus we completed the proof. □

Corollary 3.14. *Let $n, a \geq 1$, $b \geq 3$, $\mathbf{n} = (n, 1)$ and $\mathbf{a} = (a, b)$. Then $X_{\mathbf{n},\mathbf{a}}$ is not defective except for $(n, a, b) = (n, 2, 2k)$.*

Proof. In the previous theorem we proved the statement for $a, b \geq 3$. So we need only to consider the cases $a = 1, 2$. Theorem 3.5 implies that the statement is true if $a = 1$. By Remark 3.9 and Theorem 3.6, $X_{\mathbf{n},(2,b)}$ is defective if and only if b is even. Thus we completed the proof. □

Theorem 3.15. *Suppose that $T(n, m; 3, 3)$, $T(n, m; 3, 4)$ and $T(n, m; 4, 4)$ are true for any n and m. Then $T(m, n; a, b)$ is true for any $a, b \geq 3$.*

Proof. We have already shown that $T(1, m; a, b)$ is true (see Theorem 3.13). It follows from Theorem 2.9, Lemma 3.3, and Lemma 3.4 that it is sufficient to prove that $T(n, m; 3, b)$ and $T(n, m; 4, b)$ are true for every $b \geq 3$.

We first prove that $T(n, m; 3, b)$ is true for every $b \geq 3$. It has been already proved in Theorem 3.13 that $T(n, 1; 3, b)$ is true. We know by assumption that $T(n, m; 3, 3)$ and $T(n, m; 3, 4)$ are true. Thus the truth of $T(n, m; 3, b)$ immediately follows from Theorem 2.9.

We can analogously prove that $T(n, m; 4, b)$ is true for every $b \geq 3$. Indeed, $T(n, 1; 4, b)$ holds by Theorem 3.13, and $T(n, m; 4, 3)$ and $T(n, m; 4, 4)$ are true by assumption. Thus, by Theorem 2.9, $T(n, m; 4, b)$ is also true. □

4. Classification of s-defective Segre-Veronese varieties with $s \leq 4$

This section is devoted to the classification of all the defective s^{th} secant varieties of Segre-Veronese varieties with $s \in \{2, 3, 4\}$. Let $k \in \mathbb{N}$ and let $\mathbf{n} = (n_1, \ldots, n_k)$, $\mathbf{a} = (a_1, \ldots, a_k) \in (\mathbb{Z}_{\geq 0})^k \setminus \{(0, \ldots, 0)\}$. The defective s^{th} secant varieties of Segre varieties, i.e., Segre-Veronese varieties $X_{\mathbf{n},\mathbf{a}}$ with $\mathbf{a} = (1^k)$, has been completely classified for such an s in [3] and for $k \geq 3$. On the other hand it is well known that $T(n_1, n_2; 1, 1; s)$ is true except when $2 \leq s \leq n_1 \leq n_2$. We thus restrict our attention to the classification of defective s^{th} secant varieties of Segre-Veronese varieties $X_{\mathbf{n},\mathbf{a}}$ for $\mathbf{a} > (1^k)$. Let us first reformulate Lemma 2.16 (ii) as follows:

Lemma 4.1. *Suppose that $k \geq 2$ and that $s \in \{2, 3, 4\}$. If the following are satisfied:*

(i) $\mathbf{m} \geq \mathbf{n}$*;*

(ii) $\mathbf{a} > (1,1)$ *if* $k=2$*;*
(iii) $(\mathbf{n};\mathbf{a};s)$ *is subabundant and*
(iv) $T(\mathbf{n};\mathbf{a};s)$ *is true,*

then $T(\mathbf{m};\mathbf{a};s)$ *is also true.*

Proof. If $\mathbf{n}=\mathbf{m}$, there is nothing to prove. So we may assume that $\mathbf{n}<\mathbf{m}$. Let $\Omega=\{i\in\{1,\dots,k\}\mid m_i>n_i \text{ and } a_i\geq 1\}$. Then we showed that $\Omega\neq\emptyset$ in Lemma 2.16 (ii). Lemma 2.16 (ii) also says that, in order to prove this lemma, we only need to establish

$$s\leq \min_{h\in\Omega}\left\{\binom{n_h+a_h-1}{a_h-1}\prod_{i\neq h}\binom{n_i+a_i}{a_i}\right\}.$$

Without loss of generality, we may assume that

$$\binom{n_1+a_1-1}{a_1-1}\prod_{i=2}^{k}\binom{n_i+a_i}{a_i}=\min_{h\in\Omega}\left\{\binom{n_h+a_h-1}{a_h-1}\prod_{i\neq h}\binom{n_i+a_i}{a_i}\right\}$$

If $\mathbf{a}\geq(1^k)$ and $k\geq 3$, then

$$\begin{aligned}\binom{n_1+a_1-1}{a_1-1}\prod_{i=2}^{k}\binom{n_i+a_i}{a_i} &\geq \binom{n_1+a_1-1}{a_1-1}\binom{n_2+a_2}{a_2}\binom{n_3+a_3}{a_3}\\ &\geq \binom{n_1+1-1}{1-1}\binom{n_2+1}{1}\binom{n_3+1}{1}\\ &\geq 1\,(n_2+1)(n_3+1)\\ &\geq 2\cdot 2\\ &\geq s.\end{aligned}$$

Suppose now that $k=2$. If $a_1\geq 2$, then

$$\begin{aligned}\binom{n_1+a_1-1}{a_1-1}\binom{n_2+a_2}{a_2} &\geq \binom{n_1+2-1}{2-1}\binom{n_2+1}{1}\\ &\geq (n_1+1)(n_2+1)\\ &\geq 2\cdot 2\\ &\geq s.\end{aligned}$$

Similarly, if $a_1=1$ and if $a_2\geq 3$, then $\binom{n_1+a_1-1}{a_1-1}\binom{n_2+a_2}{a_2}\geq 4$. Suppose now that $(a_1,a_2)=(1,2)$. Analogously, we can immediately check that $\binom{n_1+a_1-1}{a_1-1}\binom{n_2+a_i}{a_2}\geq 3$. Now assume that $s=4$ and note that $(n_1,1;1,2;4)$ is superabundant for every positive integer n_1. Thus we may assume that $n_2\geq 2$. Then it is straightforward to see that also in this case $\binom{n_1+a_1-1}{a_1-1}\binom{n_2+a_2}{a_2}\geq 4=s$. □

For fixed $k\geq 2$, $\mathbf{a}>(1,1,0,\dots,0)$ and $s\in\{2,3,4\}$, let us consider the following partially ordered set:

$$M=\left\{\mathbf{n}\in\mathbb{N}^k\mid(\mathbf{n};\mathbf{a};s)\text{ is subabundant}\right\}.$$

Lemma 4.1 implies that, in order to prove that $T(\mathbf{n};\mathbf{a};s)$ is true for every $\mathbf{n}\in M$, it is enough to prove that $T(\mathbf{n};\mathbf{a};s)$ is true for every minimal element of M (there are only finitely many minimal elements in M). In particular, if $(1^k;\mathbf{a};s)$ is subabundant and if $T(1^k;\mathbf{a};s)$ is true, then $T(\mathbf{n};\mathbf{a};s)$ is also true for every $\mathbf{n}\in\mathbb{N}^k$.

In this case, $T(\mathbf{m};\mathbf{b};s)$ is also true for every $\mathbf{m} \in \mathbb{N}^\ell$ and for every $\mathbf{b} \in \mathbb{N}^\ell$ with $\mathbf{b} \geq \mathbf{a}$ and $\ell \geq k$, by Lemma 2.16 (i) and Lemma 4.1.

One can readily show that there are only finitely many superabundant k-tuple $(\mathbf{n};\mathbf{a};s)$ if $s \in \{2,3,4\}$ except for $(n,1;1,2;3)$, $(n,1;1,2;4)$ and $(n,1;1,3;4)$ with $n \geq 1$. Since we have already proved, in Theorem 3.5, that $T(\mathbf{n};\mathbf{a};s)$ is true for each $(\mathbf{n};\mathbf{a};s) \in \{(n,1;1,2;3),(n,1;1,2;4),(n,1;1,3;4) \mid n \geq 1\}$, we only need to show the truth of a finite number of statements to complete the classification of defective s^{th} secant varieties of Segre-Veronese varieties for the desired s.

In order to prove that $T(\mathbf{n};\mathbf{a};s) = S(\mathbf{n};\mathbf{a};s;0;0)$ is true for a given $(\mathbf{n};\mathbf{a};s)$, we apply Theorem 2.13 that allows us to reduce it to proving the truth of two statements of the forms $S(\mathbf{n}';\mathbf{a};s';0;s'')$ and $S(\mathbf{n}'';\mathbf{a};s'';0;s')$, where $(\mathbf{n}';\mathbf{a};s';0;s'')$ and $(\mathbf{n}'';\mathbf{a};s'';0;s')$ have the same abundancy. If the truth of at least one of these statements, say $S(\mathbf{n}';\mathbf{a};s';0;s'')$, is not known yet, then we apply Theorem 2.13 to $S(\mathbf{n}';\mathbf{a};s';0;s'')$. In order to prove that $T(\mathbf{n};\mathbf{a};s)$ is true, one must repeat the same process over an over until one achieves the statements that are all known to be true. This procedure is sometimes tedious to explicitly describe with words. To avoid tediousness, we will represent this process by a tree diagram as follows: Let $S(\mathbf{n};\mathbf{a};s;t;v)$ be a statement one wishes to prove to be true. Then the application of Theorem 2.13 can be represented as the following binary tree:

$$\begin{tikzcd} & \boxed{S(\mathbf{n};\mathbf{a};s;t;v)} \arrow[dl] \arrow[dr] & \\ \boxed{S(\mathbf{n}';\mathbf{a};s';t';v+s'')} & & \boxed{S(\mathbf{n}'';\mathbf{a};s'';t'';v+s')} \end{tikzcd}$$

If the statements at the leaves of the tree are identical, we draw

$$\begin{tikzcd} \boxed{S(\mathbf{n};\mathbf{a};s;t;v)} \arrow[d] \\ \boxed{2 * S(\mathbf{n}';\mathbf{a};s';t';v+s'')} \end{tikzcd}$$

instead of a usual binary tree. In this case, $(\mathbf{n};\mathbf{a};s;t;v)$ and $(\mathbf{n}';\mathbf{a};s';t';v+s'')$ should have the same abundancy.

The tree grows downward until one achieves only leaf nodes which are known to be true. By Theorem 2.13, in order to prove that the statement at the root is true, it suffices to show that the leaf statements are all true and have the same abundancy.

4.1. **Case 1:** $s = 2$.

Theorem 4.2. *$T(\mathbf{n};\mathbf{a};2)$ is true with the following exceptions:*

- *$k = 1$, $n_1 \geq 2$ and $\mathbf{a} = 2$;*
- *$\mathbf{n} = (n_1, n_2)$ and $\mathbf{a} = (1,1)$ with $2 \leq n_1 \leq n_2$.*

Proof. It is known by the Alexander-Hirschowitz theorem that if $k = 1$, then $T(\mathbf{n};\mathbf{a};2)$ fails if and only if $n_1 \geq 2$ and $a_1 = 2$. Thus we may assume that $k \geq 2$.

Suppose now that $\mathbf{a} = (1,2)$. Then $(\mathbf{n};\mathbf{a};2)$ is subabundant for every $\mathbf{n} \in \mathbb{N}^2$. Since $T(1,1;1,2;2)$ is true by Theorem 3.1, it follows from Lemma 2.16 and Lemma 4.1 that $T(\mathbf{n};\mathbf{a};2)$ are also true for all $\mathbf{n} \in \mathbb{N}^2$ and for all $\mathbf{a} \geq (1,2)$.

Note that $(1^3;1^3;2)$ is equiabundant and $T(1^3;1^3;2)$ is true. Hence $(\mathbf{n};\mathbf{a};2)$ are subabundant for all $\mathbf{n},\mathbf{a}\in\mathbb{N}^k$ with $k\geq 3$, and Lemma 2.16 and Lemma 4.1 imply that $T(\mathbf{n};\mathbf{a};2)$ is true, for any $\mathbf{n},\mathbf{a}>(1^3,0^{k-3})$. $\square$

4.2. **Case 2: $s=3$.**

Proposition 4.3. *The following statements are true:*

(i) $T(1^2,2;1^2,2;3)$;
(ii) $T(1,2,1;1^2,2;3)$.

Proof. Each statement can be reduced as in the following diagrams:

(i)

$$\begin{tikzcd} \boxed{T(1^2,2;1^2,2;3)=S(1^2,2;1^2,2;3;0;0)} \arrow[d] \arrow[dr] & \\ \boxed{S(0,1,2;1^2,2;2;0;1)=S(1,2;1,2;2;1;0)} & \boxed{S(0,1,2;1^2,2;1;0;2)=S(1,2;1,2;1;2;0)} \end{tikzcd}$$

(ii)

$$\begin{tikzcd} \boxed{T(1,2,1;1^2,2;3)=S(1,2,1;1^2,2;3;0;0)} \arrow[d] \arrow[dr] & \\ \boxed{S(0,2,1;1^2,2;2;0;1)=S(2,1;1,2;2;1;0)} & \boxed{S(0,2,1;1^2,2;1;0;2)=S(2,1;1,2;1;2;0)} \end{tikzcd}$$

One can easily check that the following are all subabundant:

$$(1^2,2;1^2,2;3),\quad (1,2,1;1^2,2;3),\quad (1,2;1,2;2;1;0),$$
$$(1,2;1,2;1;2;0),\quad (2,1;1,2;2;1;0),\quad (2,1;1,2;1;2;0).$$

By Theorem 4.2, $S(1,2;1,2;2;0;0)$ and $S(2,1;1,2;2;0;0)$ are true. Thus the following statements are also true:

$$S(1,2;1,2;2;1;0) \text{ and } S(2,1;1,2;2;1;0).$$

Likewise, $S(1,2;1,2;1;2;0)$ and $S(2,1;1,2;1;2;0)$ are true, because $S(1,2;1,2;1;0;0)$ and $S(2,1;1,2;1;0;0)$ are true. Thus we conclude that

$$T(1^2,2;1^2,2;3) \text{ and } T(1,2,1;1^2,2;3)$$

are true. $\square$

Proposition 4.4. $T(1^4;1^3,2;3)$ *is true.*

Proof. We can reduce this statement as follows:

$$\begin{tikzcd} \boxed{T(1^4;1^3,2;3)=S(1^4;1^3,2;3;0;0)} \arrow[d] \arrow[dr] & \\ \boxed{S(0,1^3;1^3,2;2;0;1)=S(1^3;1^2,2;2;1;0)} & \boxed{S(0,1^3;1^3,2;1;0;2)=S(1^3;1^2,2;1;2;0)} \end{tikzcd}$$

Theorem 4.2 implies that $S(1^3;1^2,2;2;0;0)$ is true, from which it follows that $S(1^3;1^2,2;2;1;0)$ is true. Also, it is clear that $S(1^3;1^2,2;1;0;0)$ is true, and so $S(1^3;1^2,2;1;2;0)$ is true too. Since the following have the same abundancy:

$$(1^3;1^2,2;2;1;0),\ (1^3;1^2,2;1;2;0) \text{ and } (1^4;1^3,2;3),$$

we can conclude that $T(1^4; 1^3, 2; 3)$ is true. □

Theorem 4.5. *$T(\mathbf{n}; \mathbf{a}; 3)$ is true with the following exceptions:*

- *$k = 1$, $n_1 \geq 3$ and $\mathbf{a} = 2$;*
- *$\mathbf{n} = (n_1, n_2)$ with $3 \leq n_1 \leq n_2$ and $\mathbf{a} = (1, 1)$;*
- *$\mathbf{n} = (1, 1)$ and $\mathbf{a} = (2, 2)$;*
- *$\mathbf{n} = (1, 1, n)$ with $n \geq 3$ and $\mathbf{a} = (1, 1, 1)$;*
- *$\mathbf{n} = (1, 1, 1)$ and $\mathbf{a} = (1, 1, 2)$;*
- *$\mathbf{n} = (1^4)$ and $\mathbf{a} = (1^4)$.*

Proof. Let $k = 1$. Then from the theorem of Alexander and Hirschowitz it follows that $T(\mathbf{n}; \mathbf{a}; 3)$ is false if and only if $n_1 \geq 3$ and $a_1 = 2$. Let us assume that $k \geq 2$.

The 11-tuple $(1^5; 1^5; 3)$ is subabundant and $T(1^5; 1^5; 3)$ is true (see [18] for the proof). This means that if $k \geq 5$, then $T(\mathbf{n}; \mathbf{a}; 3)$ are true for all $\mathbf{n}, \mathbf{a} \in \mathbb{N}^k$. Thus we may assume that $k \leq 4$.

Suppose that $k = 4$. In [3], it was proved that if $\mathbf{a} = (1^4)$, then there are no defective cases except for $\mathbf{n} = (1^4)$. We have proved in Proposition 4.4 that $T(1^4; 1^3, 2; 3)$ is true. This proves, by Lemma 2.16 and Lemma 4.1, that $T(\mathbf{n}; \mathbf{a}; 3)$ is true for every $\mathbf{n} \in \mathbb{N}^4$ and $\mathbf{a} \geq (1^3, 2)$. So the theorem holds if $k = 4$, and hence we may assume that $k \leq 3$.

Let $\mathbf{a} = (1^3)$. Then $T(\mathbf{n}; \mathbf{a}; 3)$ is true except for $\mathbf{n} = (1^2, n)$ with $n \geq 3$ (see [3]). So assume that $\mathbf{a} > (1^3)$. Since $T(1^3; 1, 2^2; 3)$ is true (see [8]), it follows from Lemma 2.16 and Lemma 4.1 that if $\mathbf{a} \geq (1, 2^2)$, then $T(\mathbf{n}; \mathbf{a}; 3)$ is true for every $\mathbf{n} \in \mathbb{N}^3$. Note that $(1^3; 1^2, 3; 3)$ is subabundant. Additionally, the truth of $T(1^3; 1^2, 3; 3)$ was proved by Bauer and Draisma [8]. Thus it remains only to show that $T(\mathbf{n}; 1^2, 2; 3)$ is true except for $(\mathbf{n}) = (1^3)$. It is not hard to prove that $(\mathbf{n}; 1^2, 2; 3)$ is subabundant for every $\mathbf{n} \in \mathbb{N}^3$. Since $T(1^3; 1^2, 2; 3)$ is false, we need to show that $T(1^2, 2; 1^2, 2; 3)$ and $T(1, 2, 1; 1^2, 2; 3)$ are true and we did it in Proposition 4.3 . Thus we may now assume that $k = 2$.

Suppose that $\mathbf{a} = (1, 1)$. It is known that $T(\mathbf{n}; \mathbf{a}; 3)$ holds if and only if $3 \leq n_1 \leq n_2$. Suppose that $\mathbf{a} = (1, 2)$. Then $(\mathbf{n}; \mathbf{a}; 3)$ is subabundant if $\mathbf{n} \neq (2, 1), (1, 1)$. It was already proved in [8] that $T(1, 2; 1, 2; 3)$ is true. Thus $T(\mathbf{n}; 1, 2; 3)$ is true of $\mathbf{n} \geq (1, 2)$. By Theorem 3.5, $T(n, 1; 1, 2; 3)$ with $n \geq 1$ is true, and hence $T(\mathbf{n}; 1, 2; 3)$ is true for every $\mathbf{n} \in \mathbb{N}^2$.

Let $\mathbf{a} = (1, 3)$. Then $(\mathbf{n}; \mathbf{a}; 3)$ is superabundant if and only if $\mathbf{n} = (1, 1)$. Note that $T(1, 1; 1, 3; 3)$ is true, by Theorem 3.5. It was also proved that both $T(1, 2; 1, 3; 3)$ and $T(2, 1; 1, 3; 3)$ are true (see [8]), which implies that $T(\mathbf{n}; 1, 3; 3)$ is true for any $\mathbf{n} \in \mathbb{N}^2$.

Next consider $\mathbf{a} = (1, 4)$. Clearly, $(\mathbf{n}; \mathbf{a}; 3)$ is subabundant for every $\mathbf{n} \in \mathbb{N}^2$. Since $T(1, 1; 1, 4; 3)$ is true by Theorem 3.5, $T(\mathbf{n}; 1, b; 3)$ is also true for each $b \geq 4$ and $\mathbf{n} \in \mathbb{N}^2$.

Let $\mathbf{a} = (2, 2)$. Then $T(1, 1; \mathbf{a}; 3)$ is known to be false. Let $\mathbf{n} = (1, 2)$. Then $(\mathbf{n}; \mathbf{a}; 3)$ is subabundant. Since $(\mathbf{n}; 1, 1; 3)$ is subabundant and since $T(\mathbf{n}; 1, 1; 3)$ is true, $T(\mathbf{n}; \mathbf{a}; 3)$ is also true. This also proves that $T(\mathbf{n}; \mathbf{a}; 3)$ is true if $\mathbf{n} > (1, 1)$ and if $\mathbf{a} \geq (2, 2)$. Thus it remains only to prove that $T(1^2; 2, 3; 3)$ is true, because $(1^2; 2, 3; 3)$ is subabundant. But $T(1^2; 2, 3; 3)$ is true by Theorem 3.1. Therefore we can conclude that if $k = 2$, then $T(\mathbf{n}; \mathbf{a}; 3)$ fails if and only if $\mathbf{n} = (n_1, n_2)$ with $3 \leq n_1 \leq n_2$ and $\mathbf{a} = (1, 1)$ or $\mathbf{n} = (1, 1)$ and $\mathbf{a} = (2, 2)$. □

4.3. **Case 3:** $s = 4$. Let $\mathbf{n} = (n^2, 1)$ with $n \geq 2$ and let $\mathbf{a} = (1^2, 2)$. Then $T(\mathbf{n}; \mathbf{a}; n+2)$ is known to be false by [13, Corollary 5.5]. Here we give a different, but shorter proof of the same result:

Proposition 4.6. *$T(n^2, 1; 1^2, 2; n+2)$ is false for every $n \geq 2$.*

Proof. Let $\mathbf{n} = (n^2, 1)$ and let $\mathbf{a} = (1^2, 2)$. The defectivity of $\sigma_{n+2}(X_{\mathbf{n},\mathbf{a}})$ can be proved by the existence of a certain rational normal curve in X_{2n+2} passing through generic $(n+2)$ points of $X_{\mathbf{n},\mathbf{a}}$.

For each $i \in \{1, 2, 3\}$, let π_i the canonical projection from $\mathbb{P}^{\mathbf{n}}$ to the i^{th} factor of $\mathbb{P}^{\mathbf{n}}$. Given generic points $p_1, \ldots, p_{n+2} \in \mathbb{P}^{\mathbf{n}}$, let $q_i = \pi_3(p_i) \in \mathbb{P}^1$. Since any ordered subset of $n+2$ points in general position in $\mathbb{P}^n$ is projectively equivalent to the ordered set $\{\pi_i(p_1), \ldots, \pi_i(p_{n+2})\}$ for $i \in \{1, 2\}$, there is a rational normal curve $\nu_{n,i} : \mathbb{P}^1 \to C_n \subset \mathbb{P}^n$ of degree n such that $\nu_{n,i}(q_j) = \pi_i(p_j)$ for all $j \in \{1, \ldots, n+2\}$. Let $\nu = (\nu_{n,1}, \nu_{n,2}, \mathrm{id})$ and let $C = \nu(\mathbb{P}^1)$. Then C passes through $p_1, \ldots, p_{n+2}$. The image of C under the morphism given by $\mathcal{O}(1^2, 2)$ is a rational normal curve of degree $2n+2 (= n+n+2\cdot 1)$ in $\mathbb{P}^{2n+2}$. Thus we have

$$\begin{aligned}
\dim \sigma_{2n+2}(X_{\mathbf{n},\mathbf{a}}) &\leq 2n+2+(n+2)(2n+1-1) \\
&= 2n^2+6n+2 \\
&< (n+2)(2n+2)-1 \\
&= 2n^2+6n+3,
\end{aligned}$$

and so $\sigma_{2n+2}(X_{\mathbf{n},\mathbf{a}})$ is defective. □

The above proposition only proves that if $\mathbf{n} = (n^2, 1)$ with $n \geq 2$ and $\mathbf{a} = (1^2, 2)$, then $\dim \sigma_{n+2}(X_{\mathbf{n},\mathbf{a}}) \leq 2n^2+6n+2$. Below we will show that the equality actually holds.

Proposition 4.7. *Let $\mathbf{n} = (n^2, 1)$ with $n \geq 2$ and let $\mathbf{a} = (1^2, 2)$. Then*

$$\dim \sigma_{n+2}(X_{\mathbf{n},\mathbf{a}}) = 2n^2+6n+2.$$

Proof. The statement $T(\mathbf{n}; \mathbf{a}; n+2)$ can be reduced to $S(n-1, n, 1; 1^2, 2; n+1; 0; 1)$ and $S(0, n, 1; 1^2, 2; 1; 0; n+1) = S(n, 1; 1, 2; 1; n+1; 0)$. Since $(n, 1; 1, 2; 1; n+1; 0)$ is subabundant and $S(n, 1; 1, 2; 1; 0; 0)$ is clearly true, it follows that $S(n, 1; 1, 2; 1; n+1; 0)$ is also true. We can reduce $S(n-1, n, 1; 1^2, 2; n+1; 0; 1)$ to

$$S(n-2, n, 1; 1^2, 2; n; 0; 2) \text{ and } S(n, 1; 1, 2; 1; n+1; 0).$$

We continue in this manner until we reduce to

$$S(0, n, 1; 1^2, 2; 2; 0; n) = S(n, 1; 1, 2; 2; n; 0) \text{ and } n * S(n, 1; 1, 2; 1; n+1; 0).$$

Since $(n, 1; 1, 2; 2; n-1; 0)$ is equiabundant and since $S(n, 1; 1, 2; 2; 0; 0)$ is true by Proposition 4.2, $S(n, 1; 1, 2; 2; n-1; 0)$ is also true. This proves the truth of $S(n, 1; 1, 2; 2; n; 0)$. The truth of $S(n, 1; 1, 2; 2; n; 0)$ implies that the linear subspace spanned by $\sigma_2(X_{(n,1),(1,2)})$ and n generic points coincides with $\mathbb{P}^{3(n+1)-1}$; while the truth of $S(n, 1; 1, 2; 1; n+1; 0)$ implies that the linear subspace spanned by $\sigma_1(X_{(n,1),(1,2)})$ and $n+1$ generic points has dimension $2n+2 = (n+2)+(n+1)-1$. Therefore,

$$\begin{aligned}
\dim \sigma_{n+2}(X_{\mathbf{n},\mathbf{a}}) &\geq 3n+3+n(2n+3)-1 \\
&= 2n^2+6n+2.
\end{aligned}$$

Thus we obtain $\dim \sigma_{n+2}(X_{\mathbf{n},\mathbf{a}}) = 2n^2+6n+2$. □

More generally, let $\mathbf{n} = (n, n, 1)$ with $n \geq 2$, let $\mathbf{a} = (1, 1, 2d)$ with $d \geq 1$ and let $\lfloor \frac{(2d+1)(n+1)}{2} \rfloor \leq s \leq nd + n + d$. Then $(\mathbf{n}; \mathbf{a}; s)$ is superabundant. Thus $\dim \sigma_s(X_{\mathbf{n},\mathbf{a}})$ is expected to be $(n+1)^2(2d+1) - 1$. However, there exist a form f_1 of multi-degree $(1, 0, d)$ and a form f_2 of multi-degree $(0, 1, d)$, both of which vanish at given s generic simple points. Thus the form $f = f_1 f_2$ of multi-degree $(1, 1, 2d)$ vanishes at the s generic double points. So $\dim \sigma_s(X_{\mathbf{n},\mathbf{a}}) < (n+1)^2(2d+1) - 1$, and hence $\sigma_s(X_{\mathbf{n},\mathbf{a}})$ is defective for every $\lfloor \frac{(2d+1)(n+1)}{2} \rfloor \leq s \leq nd + n + d$. It is worth mentioning that these defective cases were first found by Catalisano, Geramita and Gimigliano in [16].

Our computer experiments suggest, however, that $\sigma_s(X_{\mathbf{n},\mathbf{a}})$ is also defective for any $d(n+1)+1 \leq s \leq \lfloor \frac{(2d+1)(n+1)}{2} \rfloor - 1$. On the other hand, one can slightly modify the proof of Proposition 4.7 to show that $\sigma_s(X_{\mathbf{n},\mathbf{a}})$ is not defective if $s \leq d(n + 1)$ or if $s \geq nd + n + d + 1$. Thus it is very natural to ask the following question:

Question 4.8. *Let $\mathbf{n}$ and $\mathbf{a}$ be as above. Then is $\sigma_s(X_{\mathbf{n},\mathbf{a}})$ defective exactly when $d(n + 1) + 1 \leq s \leq d(n + 1) + n$?*

The answer to this question is known to be affirmative for $d = 1$. But, to our best knowledge, the question is still open in general.

Proposition 4.9. *Let $k \in \mathbb{N}$ and let $\mathbf{n} = (n_2, \ldots, n_k) \in \mathbb{N}^{k-1}$ and let $\mathbf{a} = (a_2, \ldots, a_k) \in \mathbb{N}^{k-1}$ with either $\mathbf{a} > (1, 1, 0, \ldots, 0)$ or $\mathbf{a} > (2, 0, \ldots, 0)$. Then $T(1, \mathbf{n}; 1, \mathbf{a}; 4)$ is true.*

Proof. Let $\mathbf{n} = (n_2, \ldots, n_k)$ and let $\mathbf{a} = (a_2, \ldots, a_k)$. The statement can be reduced as in the following diagrams:

$$\begin{tikzcd} \boxed{T(1, \mathbf{n}; 1, \mathbf{a}; 4) = S(1, \mathbf{n}; 1, \mathbf{a}; 4; 0; 0)} \arrow[d] \\ \boxed{2 * S(0, \mathbf{n}; 1, \mathbf{a}; 2; 0; 2) = 2 * S(\mathbf{n}; \mathbf{a}; 2; 2; 0)} \end{tikzcd}$$

Then $(1, \mathbf{n}; 1, \mathbf{a}; 4)$ and $(\mathbf{n}; \mathbf{a}; 2; 2; 0)$ must have the same abundancy. By Theorem 4.2, $S(\mathbf{n}; \mathbf{a}; 2; 0; 0) = T(\mathbf{n}; \mathbf{a}; 2)$ is true because of the assumption on $\mathbf{a}$. Thus $S(\mathbf{n}; \mathbf{a}; 2; 2; 0)$ is true, from which the truth of $T(1, n; 1, d; 4)$ follows. $\square$

The following is an immediate consequence of Proposition 4.9:

Corollary 4.10. *The following statements are true:*

(1) $T(1, 2; 1, b; 4)$ *are true for* $b \in \{3, 4\}$*;*
(2) $T(1, 3, 1; 1^2, 2; 4)$*;*
(3) $T(1^3; 1^2, 2; 4)$*;*
(4) $T(1^3; 1, 2^2; 4)$*;*
(5) $T(1^2, 2; 1^2, 2; 4)$*;*
(6) $T(1, 2, 1; 1^2, 2; 4)$*;*
(7) $T(1^3; 1^2, 3; 4)$.

Proposition 4.11. *The following statements are all true:*

(i) $T(2, 3, 1; 1^2, 2; 4)$*;*
(ii) $T(2^3; 1^2, 2; 4)$*;*

Proof. Each statement in this theorem can be reduced as indicated in the following diagram:

(i)

$$\begin{tikzcd}
\boxed{T(2,3,1;1^2,2;4) = S(2,3,1;1^2,2;4;0;0)} \arrow[d] \\
\boxed{2 * S(2,1,1;1^2,2;2;0;2)} \arrow[d] \\
\boxed{4 * S(2,0,1;1^2,2;1;0;3) = 4 * S(2,1;1,2;1;3;0)}
\end{tikzcd}$$

(ii)

$$\begin{tikzcd}
\boxed{T(2^3;1^2,2;4) = S(2^3;1^2,2;4;0;0)} \arrow[d] \arrow[dr] & \\
\boxed{S(1^2,2;1^2,2;2;0;2)} \arrow[d] & \boxed{S(0,2^2;1^2,2;2;0;2) = S(2^2;1,2;2;2;0)} \\
\boxed{2 * S(0,2^2;1^2,2;1;0;3) = 2 * S(2^2;1,2;1;3;0)} &
\end{tikzcd}$$

Note that the following are all subabundant:

$$(2,3,1;1^2,2;4),\ (2,1;1,2;1;3;0),\ (2^3;1^2,2;4), (2^2;1,2;2;2;0),\ (2^2;1,2;1;3;0).$$

Since $S(2^2;1,2;2;0;0)$ is true by Theorem 4.2, so is $S(2^2;1,2;2;2;0)$. Furthermore, both $S(2,1;1,2;1;0;0)$ and $S(2^2;1,2;1;0;0)$ are true. Thus $S(2,1;1,2;1;3;0)$ and $S(2^2;1,2;1;3;0)$ are also true. This means that all the statements that appear at the leaf nodes in each tree are true. Thus also $T(2,3,1;1^2,2;4)$ and $T(2^3;1^2,2;4)$ are true. □

Proposition 4.12. $T(1^4;1^3,2;4)$ *is true.*

Proof. This statement can be reduced as follows:

$$\begin{tikzcd}
\boxed{T(1^4;1^3,2;4) = S(1^4;1^3,2;4;0;0)} \arrow[d] \\
\boxed{2 * S(0,1^3;1^3,2;2;0;2) = 2 * S(1^3;1^2,2;2;2;0)}
\end{tikzcd}$$

The classification of defective second secant varieties of Segre-Veronese varieties (Theorem 4.2) implies that $S(1^3;1^2,2;2;0;0)$ is true, from which the truth of $S(1^3;1^2,2;2;2;0)$ follows. Since $(1^3;1^2,2;2;2;0)$ and $(1^4;1^3,2;4)$ have the same abundancy, we conclude that $T(1^4;1^3,2;4)$ is true. □

Theorem 4.13. *Let $k \in \mathbb{N}$ and let $\mathbf{n}, \mathbf{a} \in \mathbb{N}^k$. Then $T(\mathbf{n};\mathbf{a};4)$ is false if and only if $(\mathbf{n};\mathbf{a})$ falls into one of the following cases:*

- $(\mathbf{n};\mathbf{a}) = (n,2)$ *with* $n \geq 4$*;*
- $\mathbf{n} = (n_1,n_2)$ *with* $4 \leq n_1 \leq n_2$ *and* $\mathbf{a} = (1,1)$*;*
- $(\mathbf{n};\mathbf{a}) = (1,2;2^2)$*;*
- $(\mathbf{n};\mathbf{a}) = (2^3;1^3)$*;*

- $(\mathbf{n}; \mathbf{a}) = (1, 2, n; 1^3)$ *with* $n \geq 4$*;*
- $(\mathbf{n}; \mathbf{a}) = (2^2, 1; 1^2, 2)$.

Proof. The Alexander-Hirschowitz theorem says that if $k = 1$, then $T(\mathbf{n}; \mathbf{a}; 4)$ fails if and only if $(\mathbf{n}; \mathbf{a}) = (n, 2)$ with $n \geq 4$. Thus we may assume that $k \geq 2$.

The 11-tuple $(1^5; 1^5; 4)$ is subabundant and the truth of $T(1^5; 1^5; 4)$ has been proved by Catalisano, Geramita and Gimigliano (see [18]). This means that if $k \geq 5$, then $T(\mathbf{n}; \mathbf{a}; 4)$ is true for all $\mathbf{n}, \mathbf{a} \in \mathbb{N}^k$. Thus we may assume that $k \leq 4$.

Suppose first that $k = 4$. In [3], it was proved that there are no defective cases if $\mathbf{a} = (1^4)$. Note that $(1^3, 2; 1^4; 4)$ is equiabundant. Thus it follows from Lemma 2.16 and Lemma 4.1 that $T(\mathbf{n}; \mathbf{a}; 4)$ is true for every $\mathbf{n} \in \mathbb{N}^k$ with $\mathbf{n} \geq (1^3, 2)$ and for every $\mathbf{a} \in \mathbb{N}^k$. Moreover, we have already proved that $T(1^4; 1^3, 2; 4)$ is true (see Proposition 4.12). This proves that $T(1^4; \mathbf{a}; 4)$ is true for every $\mathbf{a} \in \mathbb{N}^4$. So the theorem holds if $k \geq 4$, and thus we may assume that $k \leq 3$.

Suppose now that $k = 3$. In [3, Theorem 4.6], $T(\mathbf{n}; 1^4; 3)$ was proved to be true except for $\mathbf{n} = (2^3)$ and $\mathbf{n} = (1, 2, n)$ with $n \geq 4$. So we may assume that $\mathbf{a} > (1^3)$. In Corollary 4.10 it is proved that $T(1^3; 1, 2^2; 4)$ and $T(1^3; 1^2, 3; 4)$ are true. Since $(1^3; 1, 2^2; 4)$ and $(1^3; 1^2, 3; 4)$ are both subabundant, it follows that $T(\mathbf{n}; \mathbf{a}; 4)$ is true for every $\mathbf{n} \in \mathbb{N}^3$ if $\mathbf{a} \geq (1^2, 3)$ or $\mathbf{a} \geq (1, 2^2)$. This means that it remains only to prove the truth of $T(\mathbf{n}; 1^2, 2; 4)$ for every $\mathbf{n} \in \mathbb{N}^3$ except for $\mathbf{n} = (2^2, 1)$. A 7-tuple $(\mathbf{n}; 1^2, 2; 4)$ is not equiabundant, but superabundant precisely when $\mathbf{n} = (1^3)$ and $(1, 2, 1)$. Additionally, we have proved that $T(2^2, 1; 1^2, 2; 4)$ is false in Proposition 4.7. Thus all we need to do is show that $T(\mathbf{n}; 1^2, 2; 4)$ are true for all $\mathbf{n} \in \{(1^3), (1, 2, 1), (1, 3, 1), (1^2, 2), (2^3), (2, 3, 1)\}$. Those statements were, however, proved to be true in Corollary 4.10 and Proposition 4.11.

Finally, assume that $k = 2$. In [8] it was already proved that $T(1^2; \mathbf{a}; 4)$ is true for every $\mathbf{a} \in \mathbb{N}^2$ and that $T(1, 2; \mathbf{a}; 4)$ is true for every $\mathbf{a} \in \mathbb{N}^2$ except for $\mathbf{a} = (2, 2)$. Since the 5-tuple $(1^2; 2, 3; 4)$ is equiabundant and the statement $T(1^2; 2, 3; 4)$ is true, we conclude, by Lemma 2.16 and Lemma 4.1, that $T(\mathbf{n}; \mathbf{a}; 4)$ is true for every $\mathbf{a} \geq (2, 3)$ and every $\mathbf{n} \in \mathbb{N}^2$. This means that to complete the proof it is enough to prove the truth of $T(\mathbf{n}; \mathbf{a}; 4)$ for $\mathbf{a} = (1, d)$ with $d \geq 2$ and for $\mathbf{a} = (2, 2)$.

Assume first that $\mathbf{a} = (2, 2)$. The 5-tuple $(2^2; 2^2; 4)$ is subabundant and the statement $T(2^2; 2^2; 4)$ is true by Example 2.15 and Remark 2.12 (ii), because the 5-tuple $(2^2; 2^2; 5)$ is also subabundant. So, by Lemma 4.1, we conclude that $T(\mathbf{n}; 2^2; 4)$ is true for every $\mathbf{n} \geq (2, 2)$.

We now consider the case $\mathbf{a} = (1, d)$. Note that $T(1^2; 1, 5; 4)$ is true and $(1^2; 1, 5; 4)$ is equiabundant. Thus, by Lemma 2.16 and Lemma 4.1, $T(\mathbf{n}; 1, d; 4)$ is also true for all $d \geq 5$ and for any $\mathbf{n} \in \mathbb{N}^2$. If $d = 4$, $(1^2; 1, 4; 4)$ and $(2, 1; 1, 4; 4)$ are the only non-subabundant 5-tuples. Thus $T(\mathbf{n}; 1, 4; 4)$ is true for every $\mathbf{n}$, because the truth of $T(\mathbf{n}; 1, 4; 4)$ was already proved to be true for every $\mathbf{n} \in \{(1^2), (2, 1), (3, 1), (1, 2)\}$. Let $d = 3$. It is straightforward to prove that $(\mathbf{n}; 1, 3; 4)$ is not subabundant if and only if $\mathbf{n} = (m, 1)$ with $m \geq 1$, and $T(m, 1; 1, 3; 4)$ is true for every $m \geq 1$, by Theorem 3.5. Furthermore, we proved in Corollary 4.9 that $T(1, 2; 1, 3; 4)$ is true. This means that $T(\mathbf{n}; 1, 3; 4)$ holds for every $\mathbf{n} \in \mathbb{N}^2$. Finally, suppose that $d = 2$. It is immediate to show that $(\mathbf{n}; 1, 2; 4)$ is not subabundant if and only if $\mathbf{n} = (1, 2)$, $\mathbf{n} = (2, 2)$ or $\mathbf{n} = (m, 1)$ with $m \geq 1$. This means that, in order to prove the truth of $T(\mathbf{n}; 1, 2; 4)$ for every $\mathbf{n} \in \mathbb{N}^2$, it is sufficient to show that $T(\mathbf{n}; 1, 2; 4)$ is true for every $\mathbf{n} \in \{(1, 2), (1, 3), (2, 2), (2, 3), (m, 1) \text{ with } m \geq 1\}$. In [2], it was proved that $T(1, n; 1, 2; 4)$ are true for $n = \{2, 3\}$ and that $T(2, 3; 1, 2; 4)$ is true. The truth of

$T(2,2;1,2;4)$ was shown in [1]. Finally $T(m,1;1,2;4)$ is true for every $m \geq 1$, by Theorem 3.5. Thus we completed the proof. $\square$

5. Conjectures

The main purpose of this section is to give a conjectural complete list of defective two-factor Segre-Veronese varieties. The first part of this section is devoted to collecting some results on defective secant varieties of Segre-Veronese varieties. To start with, we would like to consider the so-called "unbalanced" Segre-Veronese varieties.

Definition 5.1. Let $\mathbf{n} = (n_1, \ldots, n_k) \in \mathbb{N}^k$ and let $\mathbf{a} = (a_1, \ldots, a_{k-1}, 1) \in \mathbb{N}^k$.

- $(\mathbf{n}; \mathbf{a})$ is said to be *balanced* if $n_k \leq \prod_{i=1}^{k-1} \binom{n_i+a_i}{a_i} - \sum_{i=1}^{k-1} n_i$.
- $(\mathbf{n}; \mathbf{a})$ is said to be *unbalanced* if $n_k \geq \prod_{i=1}^{k-1} \binom{n_i+a_i}{a_i} - \sum_{i=1}^{k-1} n_i + 1$.

The notion of "unbalanced" was first introduced for Segre varieties (see for example [15] and [3]). Then it was extended to Segre-Veronese varieties in [17]. The following theorem was proved by Catalisano, Geramita and Gimigliano:

Theorem 5.2 ([17])**.** *Let* $\mathbf{n} = (n_1, \ldots, n_k) \in \mathbb{N}^k$ *and let* $\mathbf{a} = (a_1, \ldots, a_{k-1}, 1) \in \mathbb{N}^k$. *Suppose that* $(\mathbf{n}; \mathbf{a})$ *is unbalanced. Then* $T(\mathbf{n}; \mathbf{a}; s)$ *fails if and only if*

$$\prod_{i=1}^{k-1} \binom{n_i + a_i}{a_i} - \sum_{i=1}^{k-1} n_i < s < \min \left\{ n_k + 1, \ \prod_{i=1}^{k-1} \binom{n_i + a_i}{a_i} \right\}. \tag{9}$$

Remark 5.3. Let $\mathbf{n}$ and $\mathbf{a}$ be as given in the above theorem. Then $X_{\mathbf{n},\mathbf{a}}$ is defective if and only if Inequalities (9) have an integer solution. Since $(\mathbf{n}, \mathbf{a})$ is unbalanced, if $n_k + 1 \leq \prod_{i=1}^{k-1} \binom{n_i+a_i}{a_i}$, then (9) must have at least one integer solution.

Suppose now that $n_k + 1 > \prod_{i=1}^{k-1} \binom{n_i+a_i}{a_i}$, then (9) have an integer solution if and only if

$$\prod_{i=1}^{k-1} \binom{n_i + a_i}{a_i} - \left[\prod_{i=1}^{k-1} \binom{n_i + a_i}{a_i} - \sum_{i=1}^{k-1} n_i \right] = \sum_{i=1}^{k-1} n_i > 1.$$

This inequality holds unless $k = 2$ and $n_1 = 1$. Thus if $(\mathbf{n}, \mathbf{a})$ is unbalanced and if $(k, n_1) \neq (2, 1)$, then $X_{\mathbf{n},\mathbf{a}}$ is defective.

Many other examples of defective secant varieties of two factor Segre-Veronese varieties have also been discovered by several authors. Below we provide the list of such defective secant varieties.

Remark 5.4. For an explanation of the cases where the degree is $(1,2)$ we refer to [2, Remark 5.1]. The defective cases of degree $(2,2)$ are explained in [17, Section 3].

We are now in position to state our conjecture:

Conjecture 5.5. *Let* $\mathbf{n} = (m, n) \in \mathbb{N}^2$, *let* $\mathbf{a} = (a, b) \in \mathbb{N}^n$ *and let* $X_{\mathbf{n},\mathbf{a}}$ *be the Segre-Veronese variety* $\mathbb{P}^m \times \mathbb{P}^n$ *embedded by* $\mathcal{O}_{\mathbb{P}^m \times \mathbb{P}^n}(\mathbf{a})$. *Then* $X_{\mathbf{n},\mathbf{a}}$ *is defective if and only if* $(\mathbf{n}, \mathbf{a})$ *falls into one of the following cases:*

(a) $(\mathbf{n}; \mathbf{a}) = (m, n; a, 1)$ *is unbalanced and* $m \geq 2$.
(b) $\mathbf{n} = (1, n)$ *and* $\mathbf{a} = (2k, 2)$ *with* $k \geq 1$.
(c) $\mathbf{n} = (4, 3)$, $(2, n)$ *with* n *odd and* $\mathbf{a} = (1, 2)$.

$\mathbf{n}$	$\mathbf{a}$	s	References
$(2, 2k+1)$	$(1,2)$	$3k+2$	[23]
$(4,3)$	$(1,2)$	6	[14]
$(1,2)$	$(1,3)$	5	[21], [14]
$(1,n)$	$(2,2)$	$n+2 \leq s \leq 2n+1$	[16], [17], [12]
$(2,2)$	$(2,2)$	$7, 8$	[16], [17]
$(2,n)$	$(2,2)$	$\left\lfloor \frac{3n^2+9n+5}{n+3} \right\rfloor \leq s \leq 3n+2$	[16], [10]
$(3,3)$	$(2,2)$	$14, 15$	[16], [17]
$(3,4)$	$(2,2)$	19	[10]
$(n,1)$	$(2,2k)$	$kn+k+1 \leq s \leq kn+k+n$	[5]

(d) $\mathbf{n} = (1,2)$ *and* $\mathbf{a} = (1,3)$.
(e) $\mathbf{n} = (2,2)$, $(3,3)$, $(3,4)$ *and* $\mathbf{a} = (2,2)$.

Evidence for this conjecture was provided by the quoted results of many authors. Further evidence in support of the conjecture was obtained via computation. Theorem 3.15 suggests the following little weaker conjecture:

Conjecture 5.6. *Let* $\mathbf{n}$, $\mathbf{a}$ *and* $X_{\mathbf{n},\mathbf{a}}$ *be as given in Conjecture 5.5. If* $\mathbf{a} \geq (3,3)$, *there are no defective two-factor Segre-Veronese varieties* $X_{\mathbf{n},\mathbf{a}}$ *for all* $\mathbf{n} \in \mathbb{N}^2$.

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