SECANT VARIETIES OF SEGRE-VERONESE VARIETIES $\mathbb{P}^m \times \mathbb{P}^n$ EMBEDDED BY $\mathcal{O}(1,2)$

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Abstract. In this paper we study the dimension of secant varieties of Segre-Veronese varieties $\mathbb{P}^m \times \mathbb{P}^n$ embedded by the morphism given by $\mathcal{O}(1,2)$. Given the dimensions $m, n$, we provide two functions $g(m, n)$ and $\pi(m, n)$, such that the $s^{th}$ secant variety is nondefective, i.e. it has the expected dimension, if $s \leq g(m, n)$ or $s \geq \pi(m, n)$. Finally, we present a conjecturally complete list of defective secant varieties of such Segre-Veronese varieties.

1. Introduction

Let $X \subseteq \mathbb{P}^N$ be an irreducible non-singular variety of dimension $d$. Then the $s^{th}$ secant variety, denoted $\sigma_s(X)$, of $X$ is defined to be the Zariski closure of the union of the linear spans of $s$-tuples of points. The study of secant varieties has a long history. The interest in this subject goes back, at a minimum, to the Italian geometers at the turn of the 20th century. More recently, this topic has received renewed interest. Indeed, the past several decades have seen an interest in secant varieties cross an ever widening collection of disciplines including algebraic complexity theory [5, 16, 15], algebraic statistics [13, 12, 4], combinatorics [17, 18] as well as in algebraic geometry.

The major questions surrounding secant varieties center around finding invariants of those objects such as dimension. A simple dimension count suggests that the expected dimension of $\sigma_s(X)$ is $\min \{N, s(d+1) - 1\}$. We say that $X$ has a defective $s^{th}$ secant variety if $\sigma_s(X)$ does not have the expected dimension. In particular, $X$ is called defective if $X$ has a defective $s^{th}$ secant variety for some $s$. For instance, the Veronese surface $X$ in $\mathbb{P}^5$ is defective, because the dimension of $\sigma_2(X)$ is four while its expected dimension is five. A well-known classification of the defective Veronese varieties was completed in a series of papers by Alexander and Hirschowitz [3, 6]. There are corresponding conjectural complete lists of defective Segre varieties and Grassmann varieties. The case of Segre-Veronese varieties is less well-understood, although many efforts have been already made to complete the list of such varieties [8, 11, 10, 9, 7]. Even the classification of defective two-factor Segre-Veronese varieties is far from complete.

In order to classify defective Segre-Veronese varieties, it is a crucial step to prove the existence of a large family of non-defective such varieties. One of the powerful tools to establish non-defectiveness of large classes of Segre-Veronese varieties is the inductive approach based on specialization techniques which consist in placing a certain number of points on a chosen divisor. For a given $n = (n_1, \ldots, n_k) \in \mathbb{N}^k$, we write $\mathbb{P}^n$ for $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. Let $X_{n,a}$ be the Segre-Veronese variety obtained from $\mathbb{P}^n$ embedded by the morphism given by $\mathcal{O}(a)$ with $a = (a_1, \ldots, a_k) \in \mathbb{N}^k$. As we shall see in Section 2 the problem of determining the dimension of $\sigma_s(X_{n,a})$ is equivalent to the problem of determining the Hilbert function $h_{\text{pw}}(Z, a)$ of a collection $Z$ of $s$ general double points in $\mathbb{P}^n$, i.e.,

$$h_{\text{pw}}(Z, a) = \dim H^0(\mathbb{P}^n, \mathcal{O}(a)) - \dim H^0(\mathbb{P}^n, \mathcal{I}_Z(a)).$$

Suppose that $a_k \geq 2$. Denote by $n'$ and $a'$ the $k$-tuples $(n_1, n_2, \ldots, n_k - 1)$ and $(a_1, a_2, \ldots, a_k - 1)$ respectively. Given an $\mathbb{P}^{n'} \subset \mathbb{P}^n$, we have an exact sequence

$$0 \to \mathcal{I}_Z(a') \to \mathcal{I}_Z(a) \to \mathcal{I}_{Z \cap \mathbb{P}^{n'}(a)} \to 0,$$
where \( \tilde{Z} \) is the residual scheme of \( Z \) with respect to \( \mathbb{P}^{n'} \). This restriction exact sequence gives rise to the so-called Castelnuovo inequality
\[
h_{\mathbb{P}^n}(Z, a) \geq h_{\mathbb{P}^n}(\tilde{Z}, a') + h_{\mathbb{P}^{n'}}(Z \cap \mathbb{P}^{n'}, a').
\]
Thus, by semicontinuity, we can conclude that
- if \( h_{\mathbb{P}^n}(\tilde{Z}, a') \) and \( h_{\mathbb{P}^{n'}}(Z \cap \mathbb{P}^{n'}, a') \) have the expected values and
- if the degrees of \( \tilde{Z} \) and \( Z \cap \mathbb{P}^{n'} \) are both less than or both greater than \( \dim H^0(\mathbb{P}^n, \mathcal{O}(a')) \) and \( \dim H^0(\mathbb{P}^{n'}, \mathcal{O}(a)) \) respectively,
then \( h_{\mathbb{P}^n}(Z, a) \) also has the expected value. This enables one to check whether or not \( \sigma_s(X_{n, a}) \) has the expected dimension by induction on \( n \) and \( a \).

In order to apply the inductive argument we need some initial steps regarding small degrees or small dimensions. Particularly interesting is the case of two-factor Segre-Veronese varieties of degree \( a = (1, 2) \), which can be viewed as the base case of the induction. For these Segre-Veronese varieties the usual specialization technique cannot be applied because it would involve secant varieties of two-factor Segre varieties, most of which are known to be defective. Thus one needs to develop an \textit{ad hoc} approach to get results in this situation.

This paper is devoted to studying secant varieties of Segre-Veronese varieties \( \mathbb{P}^m \times \mathbb{P}^n \) embedded by the morphism given by \( \mathcal{O}(1, 2) \). Let
\[
q(m, n) = \left\lfloor \frac{(m+1)(n+2)}{2} \right\rfloor.
\]
Our main goal is to prove the following theorem:

**Theorem 1.1.** Let \( n = (m, n) \) and let \( a = (1, 2) \). If \( n \) is sufficiently large, then \( \sigma_s(X_{n, a}) \) has the expected dimension for any \( s \leq q(m, n) \).

To prove Theorem 1.1, it suffices to show that \( \sigma_{q(m, n)}(X_{n, a}) \) has the expected dimension. To show this, we define the following function
\[
s(m, n) = \begin{cases} 
(m+1) \lfloor \frac{n}{2} \rfloor - \frac{(m-2)(m+1)}{2} & \text{if } n \text{ is even;} \\
(m+1) \lfloor \frac{n}{2} \rfloor - \frac{(m-3)(m+1)}{2} & \text{if } m \text{ and } n \text{ are odd;} \\
(m+1) \lfloor \frac{n}{2} \rfloor - \frac{(m-3)(m+1)+1}{2} & \text{if } m \text{ is even and } n \text{ is odd,}
\end{cases}
\]
we prove that if \( m \leq n + 2 \), then \( \sigma_{s(m, n)}(X_{n, a}) \) has the expected dimension, and we show that \( s(m, n) = q(m, n) \) for a sufficiently large \( n \). The proof is by double induction on \( m \) and \( n \). More precisely, we will prove the following two statements:

(i) Let \( n = (n+1, n) \). The secant variety \( \sigma_{s(n+1, n)}(X_{n, a}) \) has the expected dimension. Note that the case \( n = (n+2, n) \) is trivial since \( s(n+2, n) = 0 \).

(ii) Let \( n' = (m, n-2) \) and \( n = (m, n) \). If \( \sigma_{s(m, n-2)}(X_{n', a}) \) has the expected dimension, then \( \sigma_{s(m, n)}(X_{n, a}) \) also has the expected dimension.

In order to prove (i), we will discuss an inductive approach that specializes a certain number of points on a subvariety of \( \mathbb{P}^m \times \mathbb{P}^n \) of the form \( \mathbb{P}^{m'} \times \mathbb{P}^n \) (see Section 2). This approach was successfully applied to study secant varieties of Segre varieties (see for example [2]).

The proof of (ii) relies on a specialization technique which allows to place a certain number of points on a two-codimensional subvariety of \( \mathbb{P}^m \times \mathbb{P}^n \) of the form \( \mathbb{P}^m \times \mathbb{P}^{n-2} \) (see Section 3). This approach can be regarded as a modification of the approach introduced in [6], that simplifies the proof of the Alexander-Hirschowitz theorem, in the case of Veronese varieties of degree 3. We also would like to stress that the same approach was extended to secant varieties of Grassmannians of planes in [1].

In Section 4, we will apply the same techniques to prove the following theorem:
Theorem 1.2. Let $n$ and $a$ be as given in Theorem 1.1 and let
\[
\tilde{s}(m,n) = \begin{cases} 
(m+1) \left\lfloor \frac{n}{2} \right\rfloor + 1 & \text{if } n \text{ is even;} \\
(m+1) \left\lfloor \frac{n}{2} \right\rfloor + 3 & \text{otherwise.}
\end{cases}
\]
Then $\sigma_s(X_{n,a})$ has the expected dimension for any $s \geq \tilde{s}(m,n)$.

In Section 5, we will give a conjecturally complete list of defective secant varieties of $X_{m,n}$. Evidence for the conjecture was provided by results in [11, 8]. Further evidence in support of the conjecture was obtained via the computational experiments we carried out. Thus the first part of this section will be devoted to explaining these experiments, which was done with the computer algebra system Macaulay2 developed by Dan Grayson and Mike Stillman [14]. All the Macaulay2 scripts needed to make these computations are available at http://www.webpages.uidaho.edu/~abo/programs.html.

2. Splitting Theorem

Let $V$ be an $(m+1)$-dimensional vector space over $\mathbb{C}$ and let $W$ be an $(n+1)$-dimensional vector space over $\mathbb{C}$. For simplicity, we write $\mathbb{P}^{m,n}$ for $\mathbb{P}^m \times \mathbb{P}^n = \mathbb{P}(V) \times \mathbb{P}(W)$. The Segre-Veronese variety $\mathbb{P}^{m,n}$ embedded by the morphism $\nu_{1,d}$ given by $\mathcal{O}(1,d)$ will be denoted by $X_{m,n}$. Let $T_p(X_{m,n})$ be the affine cone over the tangent space $T_p(X_{m,n})$ to $X_{m,n}$ at a point $p \in X_{m,n}$.

For each $p \in X_{m,n}$, there are $u \in V \setminus \{0\}$ and $v \in W \setminus \{0\}$ such that $p = [u \otimes v^d] \in \mathbb{P}(V \otimes S_d(W))$. In this case, $p$ can be identified with $([u],[v]) \in \mathbb{P}^{m,n}$. Thus $p$ also denotes $([u],[v])$. Let $p = [u \otimes v^d] \in X_{m,n}$. Then $T_p(X_{m,n}) = V \otimes v^d + u \otimes v^{d-1}W$. We denote by $Y_p(X_{m,n})$ (or just by $Y_p$) the $(m+1)$-dimensional subspace $V \otimes v^d$ of $V \otimes S_d(W)$.

Definition 2.1. Let $p_1, \ldots, p_s, q_1, \ldots, q_t$ be general points of $X_{m,n}$ and let $U_{m,n}$ be the subspace of $V \otimes S_d(W)$ spanned by $\sum_{i=1}^s T_{p_i}(X_{m,n})$ and $\sum_{j=1}^t Y_{q_j}(X_{m,n})$. Then $U_{m,n}$ is expected to have dimension
\[
\min \left\{ s(m+n+1) + t(m+1), (m+1)\left\lfloor \frac{n}{d} \right\rfloor \right\},
\]
We say that $S(m,n;1,d; s; t)$ is true if $U_{m,n}$ has the expected dimension. For simplicity, we denote $S(m,n;1,d; s; t)$ by $T(m,n;1,d; s; t)$.

Remark 2.2. Let $p_1, \ldots, p_s, q_1, \ldots, q_t$ be general points of $X_{m,n}$ and let $\sigma_s(X_{m,n})$ be the $s$th secant variety of $X_{m,n}$. Then the vector space $U_{m,n}$ can be thought of as the affine cone over the tangent space to the join $J(\mathbb{P}(Y_{q_1}), \ldots, \mathbb{P}(Y_{q_t}), \sigma_s(X_{m,n}))$ of $\mathbb{P}(Y_{q_1}), \ldots, \mathbb{P}(Y_{q_t})$ and $\sigma_s(X_{m,n})$ at a general point on the linear subspace spanned by $p_1, \ldots, p_s, q_1, \ldots, q_t$. Thus $S(m,n;1,d; s; t)$ is true if and only if $J(\mathbb{P}(Y_{q_1}), \ldots, \mathbb{P}(Y_{q_t}), \sigma_s(X_{m,n}))$ has the expected dimension. In particular, $\sigma_s(X_{m,n})$ has the expected dimension if and only if $S(m,n;1,d; s; t)$ is true.

Remark 2.3. Let $N = (m+1)(\frac{n+d}{d})$. Then $H^0(\mathbb{P}^{m,n}, \mathcal{O}(1,d))$ can be identified with the set of hyperplanes in $\mathbb{P}^{N-1}$. Since the condition that a hyperplane $H \subset \mathbb{P}^N$ contains $T_p(X_{m,n})$ is equivalent to the condition that $H$ intersects $X_{m,n}$ in the first infinitesimal neighborhood of $p$, the elements of $H^0(\mathbb{P}^{m,n}, T_{p^\nu(1,d)})$ can be viewed as hyperplanes containing $T_p(X_{m,n})$. Let $q \in X_{m,n}$. A similar argument shows that the elements of $H^0(\mathbb{P}^{m,n}, \mathcal{I}_{q^2|\mathbb{P}(Y_q)}(1,d))$ can be identified with hyperplanes containing $Y_q$, where $q^2|\mathbb{P}(Y_q)$ is a zero-dimensional subscheme of $X_{m,n}$ of length $m+1$.

Let $p_1, \ldots, p_s, q_1, \ldots, q_t \in X_{m,n}$ and let $Z = \{p_1^2, \ldots, p_s^2, q_1^2|\mathbb{P}(Y_q_1), \ldots, q_t^2|\mathbb{P}(Y_q_t)\}$. Recall that Terracini’s lemma says that the linear subspace spanned by $T_{p_1}(X_{m,n}), \ldots, T_{p_s}(X_{m,n})$ is the tangent space to $\sigma_s(X_{m,n})$ at a general point on the linear subspace spanned by $p_1, \ldots, p_s$. Thus implies that $\dim J(\mathbb{P}(Y_{q_1}), \ldots, \mathbb{P}(Y_{q_t}), \sigma_s(X_{m,n}))$ equals to the Hilbert function $h_{\mathbb{P}^{m,n}}(Z(1,d))$ of $Z$, i.e.,
\[
h_{\mathbb{P}^{m,n}}(Z(1,d)) = \dim H^0(\mathbb{P}^{m,n}, \mathcal{O}(1,d)) - \dim H^0(\mathbb{P}^{m,n}, \mathcal{I}_{Z(1,d)}).
\]
In particular,
\[
h_{\mathbb{P}^{m,n}}(Z(1,d)) = \min \{ s(m+n+1) + t(m+1), N \}.
\]
if and only if \( S(m, n; 1, d; s; t) \) is true.

**Definition 2.4.** A six-tuple \((m, n; 1, d; s; t)\) is called **subabundant** (resp. **superabundant**) if
\[
s(m + n + 1) + t(m + 1) \leq (m + 1) \binom{n + d}{d} \text{ (resp. } \geq \).
\]

We say that \((m, n; 1, d; s; t)\) is **equiabundant** if it is both subabundant and superabundant. We write statements such as \((m, n; 1, d; s)\) is subabundant (resp. superabundant, resp. equiabundant) when we really mean that \((m, n; 1, d; s; 0)\) is subabundant (resp. superabundant, resp. equiabundant).

**Remark 2.5.** Given two vectors \((s, t)\) and \((s', t')\), we say that \((s, t) \geq (s', t')\) if \(s \geq s'\) and \(t \geq t\). Suppose that \(S(m, n; 1, 2; s; t)\) is true and that \((m, n; 1, 2; s; t)\) is subabundant (resp. superabundant). Then \(S(m, n; 1, 2; s; t)\) is true for any choice of \(s'\) and \(t'\) with \((s, t) \geq (s', t')\) (resp. with \((s, t) \leq (s', t')\)).

**Remark 2.6.** Suppose that \(m = 0\). We make the following two simple remarks:

(i) Let \(q \in X_{0,m}\). Then \(P(Y_q(X_{0,n}))\) is just \(q\) itself. Thus if \(q_1, \ldots, q_t\) are general points of \(X_{0,n}\) and if \((0, n; 1, d; s; t)\) is subabundant, then \(S(0, n; 1, d; s; t)\) is true if and only if \(T(0, n; 1, d; s)\) is true.

(ii) If \(S(0, n; 1, d; n + 1; 0)\) is true, then \((0, n; 1, d; s)\) is subabundant and if \(s \geq n + 1\), then \(S(0, n; 1, d; s)\) is true.

**Theorem 2.7.** Let \(m = m' + m'' + 1\) and let \(s = s' + s''\). If \((m', n; 1, d; s'; s'' + t)\) and \((m'', n; 1, d; s'''; s'' + t)\) are subabundant (resp. superabundant, resp. equiabundant) and if \(S(m', n; 1, 2; s' + s'' + t)\) and \(S(m, n; 1, d; s'''; s'' + t)\) are true, then \((m, n; 1, d; s; t)\) is subabundant (resp. superabundant, resp. equiabundant) and \(S(m, n; 1, d; s; t)\) is true.

**Proof.** Here we only prove the theorem for the case where \((m', n; 1, d; s'; s'' + t)\) and \((m'', n; 1, d; s'''; s'' + t)\) are both subabundant, because the remaining cases can be proved in a similar manner. Let \(V'\) and \(V''\) be subspaces of \(V\) of dimensions \(m' + 1\) and \(m'' + 1\) respectively. Suppose that \(V\) is the direct sum of \(V'\) and \(V''\). Let \(p = [u \otimes v^d] \in X_{m,n}\). If \(u \in V''\), then we have
\[
T_p(X_{m,n}) = V \otimes v^d + u \otimes v^{d-1}W = (V' \otimes v^d + u \otimes v^{d-1}W) \oplus (V'' \otimes v^d) = T_{p'}(X_{m',n}) \oplus Y_{p''}(X_{m'',n}),
\]
for some \(p' \in X_{m',n}\) and \(p'' \in X_{m'',n}\). Similarly, one can prove that if \(u \in V''\), then \(T_p(X_{m,n}) = Y_{p'}(X_{m',n}) \oplus T_{p''}(X_{m'',n})\) for some \(p' \in X_{m',n}\).

Let \(q = [u' \otimes v'^2] \in X_{m,n}\). Then theorems exist \(q' \in X_{m',n}\) and \(q'' \in X_{m'',n}\) such that
\[
Y_{q'}(X_{m,n}) = V \otimes v'^d = (V' \otimes v'^d) \oplus (V'' \otimes v'^d) = Y_{q'}(X_{m',n}) \oplus Y_{q''}(X_{m'',n}).
\]

Thus one can conclude that \(U_{m,n} \simeq U_{m',n} \oplus U_{m'',n}\). By assumption, \(\dim U_{m',n} = s'(m' + n + 1) + (s'' + t)(m' + 1)\) and \(\dim U_{m'',n} = s''(m'' + n + 1) + (s' + t)(m'' + 1)\). Thus \(\dim U_{m,n} = \dim U_{m',n} + \dim U_{m'',n} = s(m + n + 1) + t(m + 1) \leq (m + 1) \binom{n + d}{d} + (m + 1) \binom{n + d}{d} = (m + 1) \binom{n + d}{d}\), and hence \((m, n; 1, d; s; t)\) is subabundant and \(S(m, n; 1, d; s; t)\) is true.

We will discuss examples to illustrate how to use Theorem 2.7 below.

**Example 2.8.** In this example, we apply Theorem 2.7 to prove that \(T(2, 2; 1, 2; s)\) is true for every \(s \leq 3\). Note that \((2, 2; 1, 2; s)\) is subabundant. Thus it suffices to show that \(T(2, 2; 1, 2; 3)\) is true. Since \((1, 2; 1, 2; 2; 1)\) and \((0, 2; 1, 2; 1; 2)\) are both subabundant, one can reduce \(T(2, 2; 1, 2; 3)\) to \(S(1, 2; 1, 2; 2; 1)\) and \(S(0, 2; 1, 2; 1; 2)\). The statement \(S(1, 2; 1, 2; 2; 1)\) can be reduced to twice \(S(0, 2; 1, 2; 1; 2)\). This means that \(T(2, 2; 1, 2; 3)\) is reduced to triple \(S(0, 2; 1, 2; 1; 2)\). Clearly \(S(0, 2; 1, 2; 1; 0)\) is true, and so is \(S(0, 2; 1, 2; 1; 2)\). Hence we completed the proof.
Example 2.9. We now want to prove that $T(n+1, n; 1, 2; s)$ is true for any $s \leq \left\lfloor \frac{n+1}{2} \right\rfloor + 1$. Note that $(n+1, n; 1, 2; s)$ is subabundant for such an $s$. Thus it is sufficient to prove that $T(n+1, n; 1, 2; s)$ is true if $s = \left\lfloor \frac{n+1}{2} \right\rfloor + 1$.

First suppose that $n$ is even, i.e., $n = 2k$ for some integer $k$. Then $s = k + 1$. Since $(2k, 2k; 1, 2; k, 1)$ and $(0, 2k; 1, 2; 1; k)$ are both subabundant, $T(2k+1, 2k; 1, 2; k+1) = S(2k+1, 2k : 1, 2; k+1; 0)$ can be reduced to $S(2k, 2k; 1, 2; k; 1)$ and $S(0, 2k; 1, 2; 1; k)$. For the same reason, $S(2k, 2k; 1, 2; k; 1)$ can be reduced to $S(2k-1, 2k; 1; 2; k-1; 2)$ and $S(0, 2k; 1, 2; 1; k)$. That means $T(2k+1, 2k; 1, 2; k+1)$ is now reduced to $S(2k-1, 2k; 1; 2; k-1; 2)$ and twice $S(0, 2k; 1, 2; 1; k)$ (we will denote it by $2 \ast S(0, 2k; 1, 2; 1; k)$). We can repeat the same process $k-2$ times to reduce to $T(2k+1, 2k; 1, 2; 1; k)$ and $(k+1) \ast S(0, 2k; 1, 2; 1; k)$.

The statement $S(k, 2k; 1, 2; 0; k+1)$ can be reduced to $S(k-1, 2k; 1, 2; 0; k+1)$ and $S(0, 2k; 1, 2; 0; k+1)$.

$S(k-1, 2k; 1, 2; 0; k+1)$ can be reduced to $S(k-2, 2k; 1, 2; 0; k+1)$ and $S(0, 2k; 1, 2; 0; k+1)$. Repeating the same process $k-2$ times, we can reduce $S(k, 2k; 1, 2; 0; k+1)$ to $(k+1) \ast S(0, 2k; 1, 2; 0; k+1)$.

Recall that $(0, 2k; 1, 2; 1; k)$ and $(0, 2k; 1, 2; 0; k+1)$ are subabundant. Thus $S(0, 2k; 1, 2; 1; k)$ and $S(0, 2k; 1, 2; 0; k+1)$ are true because $S(0, 2k; 1, 2; 1; 0)$ and $S(0, 2k; 1, 2; 0; 0)$ are true. This implies that $T(2k+1, 2k; 1, 2; k+1)$ is true.

In the same way, we can also prove that $T(n+1, n; 1, 2; s)$ is true if $n$ is odd. Indeed, $T(2k+2, 2k+1; 1, 2; k+2)$ can be reduced to $(k+2) \ast S(0, 2k+1; 1, 2; 1; k+1)$ and $(k+1) \ast S(0, 2k+1; 1, 2; 0; k+2)$. Since $S(0, 2k+1; 1, 2; 1; k+1)$ and $S(0, 2k+1; 1, 2; 0; k+2)$ are true, so is $T(2k+2, 2k+1; 1, 2; k+2)$.

As immediate consequences of Theorem 2.7, we can prove the following two propositions:

Proposition 2.10. $T(m, n; 1, 2; s)$ is true if $s \leq m + 1$ and $m \leq \binom{n+1}{2}$ or if $s \geq (m+1)(n+1)$.

Proof. We first prove that $T(m, n; 1, 2; s)$ is true for any $s \leq m + 1$. Note that $(m, n; 1, 2; s)$ is subabundant for such an $s$, it is enough to prove that $T(m, n; 1, 2; m+1)$ is true. This statement can be reduced to $(m+1) \ast S(0, n; 1, 2; 1; m)$. Since $m \leq \binom{n+1}{2}$,

$$m + m \geq \binom{n+2}{2}.$$ 

Thus $(0, n; 1, 2; 1; m)$ is subabundant. Since $S(n; 1, 2; 1; 0)$ is true, so is $S(0, n; 1, 2; 1; m)$. This implies that $T(m, n; 1, 2; m+1)$ is true.

To show that $T(m, n; 1, 2; s)$ is true for any $s \geq (m+1)(n+1)$, it is enough to prove that $T(m, n; 1, 2; (m+1)(n+1))$ is true. Note that $(m, n; 1, 2; (m+1)(n+1))$ is superabundant. The statement can be reduced to $(m+1) \ast S(0, n; 1, 2; (m+1)(n+1)n)$. Since $(0, n; 1, 2; (m+1); 0; 0)$ is superabundant and $T(0, n; 1, 2; (m+1))$ is true, $(0, n; 1, 2; (m+1); (m+1)n)$ is superabundant and $S(0, n; 1, 2; (m+1); (m+1)n)$ is true. Thus $T(m, n; 1, 2; (m+1)(n+1))$ is true.

In the next section, we will use a different technique to improve the bounds as given in Proposition 2.10.

Proposition 2.11. Suppose that $m \geq 1$ and $d \geq 3$. Let $\ell = \left\lfloor \frac{n+1}{m+n+1} \right\rfloor$ and let $h = \left\lfloor \frac{n+1}{n+1} \right\rfloor$.

(i) $T(m, n; 1, d; s)$ is true for any $s \leq \ell(m+1)$.

(ii) If $(n, d) \neq (2, 4), (3, 4), (4, 3), (4, 4)$ and if $s \geq h(m+1)$, then $T(m, n; 1, d; s)$ is true.

(iii) If $(n, d) = (2, 4), (3, 4), (4, 3)$ or $(4, 4)$, then $T(m, n; 1, d; s)$ is true for any $s \geq (h+1)(m+1)$.

Proof. (i) Suppose that $s = \ell(m+1)$. Then $T(m, n; 1, d; s)$ can be reduced to $(m+1) \ast S(0, n; 1, d; \ell \ell m)$. Since

$$\ell(m+1) + \ell m = \ell(m+n+1) \leq \frac{(n+d)}{m+n+1} (m+n+1) = \left(\frac{n+d}{d}\right),$$
(0; n; 1; d; ℓ; ℓm) is subabundant (this implies that (m; n; 1; d; s) is subabundant too). Furthermore, since ℓ < \left\lfloor \frac{n+d}{n+1} \right\rfloor, S(0; n; 1; d; ℓ; 0) is true by the Alexander-Hirschowitz theorem. Thus S(0; n; 1; d; ℓ; ℓm) is true, which implies that T(m; n; 1; d; s) is true.

(ii) Let s = h(m + 1). Then (m; n; 1; d; s) is clearly superabundant. The statement T(m; n; 1; d; s) can be reduced to (m + 1) ∗ S(0; n; 1; d; h; hm). Suppose that n \neq 3, 4. Then the Alexander-Hirschowitz theorem says that S(0; n; 1; d; h; 0) is true, and so is S(0; n; 1; d; h; hm).

(iii) Suppose that (n; d) = (2, 4), (3, 4), (4, 3) or (4, 4). Then S(0; n; 1; d; h + 1; 0) is true by the Alexander-Hirschowitz theorem, and thus S(0; n; 1; d; h; (h + 1)m) is also true. Therefore the same argument as above proves that T(m; n; 1; d; s) is true if s = (h + 1)(m + 1).

3. Segre-Veronese varieties \( \mathbb{P}^m \times \mathbb{P}^n \) embedded by \( O(1, 2) \): Subabundant Case

Let \( V \) be an \((m + 1)\)-dimensional vector space over \( \mathbb{C} \) with basis \( \{e_0, \ldots, e_m\} \) and let \( W \) be an \((n + 1)\)-dimensional vector space over \( \mathbb{C} \) with basis \( \{f_0, \ldots, f_m\} \). We denote by \( X_{m,n} \) the Segre-Veronese variety \( \mathbb{P}^{m,n} \) embedded by the morphism \( v_{1,2} \) given by \( O(1, 2) \).

**Definition 3.1.** Let \( k = \left\lfloor \frac{m}{2} \right\rfloor \) and let

\[
\mathbf{s} = \begin{cases} 
(m + 1)k - \frac{(m-2)(m+1)}{2} & \text{if } n \text{ is even;} \\
(m + 1)k - \frac{(m-3)(m+1)}{2} & \text{if } m \text{ and } n \text{ are odd;} \\
(m + 1)k - \frac{(m-3)(m+1)+1}{2} & \text{if } m \text{ is even and if } n \text{ is odd.}
\end{cases}
\]

Note that \( \mathbf{s} = 0 \) if \( m = n + 2 \).

The goal of this section is to prove that \( T(m; n; 1; 2; s) \) is true for any \( s \leq \mathbf{s} \) if \( m \leq n + 2 \). Since \((m; n; 1; 2; s)\) is subabundant, it is sufficient to prove that \( T(m; n; 1; 2; \mathbf{s}) \) is true. The proof is by double induction on \( m \) and \( n \). It is obvious that \( T(m; m - 2; 1; 2; 0) \) is true. We have already proved that if \( s = \left\lfloor \frac{m}{2} \right\rfloor + 1 \), then \( T(m; m - 1; 1; 2; \mathbf{s}) \) is true (see Example 2.9). Thus it remains only to show that if \( T(m; n - 2; 1; 2; \mathbf{s} - (m + 1)) \) is true, then so is \( T(m; n; 1; 2; \mathbf{s}) \).

Let \( U_L \) be a two-codimensional subspace of \( W \) and let \( L = \mathbb{P}(V) \times \mathbb{P}(U_L) \). Note that if \( p \) is a point of \( v_{1,2}(L) \), then the affine cone \( T_p(X_{m,n}) \) over the tangent space to \( X_{m,n} \) at \( p \) modulo \( V \otimes S_2(U_L) \) has dimension \((m + n + 1) - (m + n - 2 + 1) = 2\).

**Definition 3.2.** Let \( k \) and \( \mathbf{s} \) be the integers as given in Definition 3.1, let \( p_1, \ldots, p_{m-2(m+1)} \) be general points of \( L \), let \( q_1, \ldots, q_{m+1} \) be general points of \( \mathbb{P}^{m,n} \setminus L \) and let \( V_{m,n} \) be the vector space \( \langle V \otimes S_2(U_L), \sum_{i=1}^{m-2(m+1)} T_{p_i}(X_{m,n}), \sum_{i=1}^{m+1} T_{q_i}(X_{m,n}) \rangle \). Then the following inequality holds:

\[
\dim V_{m,n} \leq (m + 1)\left\lfloor \frac{n}{2} \right\rfloor + 2[\mathbf{s} - (m + 1)] + (m + 1)(m + 1) + 1
\]

\[
= \begin{cases} 
(m + 1)\left\lfloor \frac{n+2}{2} \right\rfloor & \text{if } n \text{ is even or if } m \text{ and } n \text{ are odd;} \\
(m + 1)\left\lfloor \frac{n+2}{2} \right\rfloor - 1 & \text{if } m \text{ is even and if } n \text{ is odd.}
\end{cases}
\]

We say \( R(m, n) \) is true if the equality holds. Remark 2.3 implies that \( R(m, n) \) is true if and only if

\[
\dim H^0(\mathbb{P}^{m,n}, \mathcal{I}_{Z \cup L}(1, 2)) = \begin{cases} 
0 & \text{if } n \text{ is even or if } m \text{ and } n \text{ are odd;} \\
1 & \text{if } m \text{ is even and if } n \text{ is odd,}
\end{cases}
\]

where \( Z = \{p_1^2, \ldots, p_{s-2(m+1)}^2, q_1^2, \ldots, q_{m+1}^2\} \).

**Proposition 3.3.** If \( R(m, n) \) is true and if \( T(m; n - 2; 1; 2; \mathbf{s} - (m + 1)) \) is true, then \( T(m; n; 1; 2; \mathbf{s}) \) is true.
Proof. Let $k = \left\lceil \frac{n}{2} \right\rceil$, let $p_1, \ldots, p_s \in \mathbb{P}^{m,n}$ and let $Z = \{p^1, \ldots, p^2\}$. Then

$$\dim H^0(\mathbb{P}^{m,n}, \mathcal{I}_Z(1,2)) \geq \begin{cases} (m+1) + \frac{(m-2)(m+1)^2}{2} & \text{if } n \text{ is even;} \\ 3(m+1) + \frac{(m-3)(m+1)(m+2)}{2} & \text{if } m \text{ and } n \text{ are odd;} \\ k + 3(m+1) + \frac{(m-3)(m+1)(m+2)+(m+2)}{2} & \text{otherwise.} \end{cases}$$

Suppose that $p_1, \ldots, p_{s-(m+1)} \in L$ and that $p_{s-m}, \ldots, p_s \in \mathbb{P}^{m,n} \setminus L$. Let $Z = \{p^1, \ldots, p^2\}$. Let $Z' = Z \cap L = \{p^1', \ldots, p^2_{-(m+1)}\}$. Then we have the following short exact sequence:

$$0 \to \mathcal{I}_{Z_L}(1,2) \to \mathcal{I}_Z(1,2) \to \mathcal{I}_{Z'}(1,2) \to 0.$$ 
Taking cohomology, we have

$$0 \to H^0(\mathbb{P}^{m,n}, \mathcal{I}_{Z_L}(1,2)) \to H^0(\mathbb{P}^{m,n}, \mathcal{I}_Z(1,2)) \to H^0(L, \mathcal{I}_{Z'}(1,2)).$$

Thus we must have

$$\dim H^0(\mathbb{P}^{m,n}, \mathcal{I}_Z(1,2)) \leq \dim H^0(\mathbb{P}^{m,n}, \mathcal{I}_{Z_L}(1,2)) + \dim H^0(L, \mathcal{I}_{Z'}(1,2)).$$

Since $R(m, n)$ and $T(m, n; 1, 2; s - (m + 1))$ are true, we have

$$\dim H^0(\mathbb{P}^{m,n}, \mathcal{I}_Z(1,2)) \leq \begin{cases} \dim H^0(L, \mathcal{I}_{Z'}(1,2)) + 1 & \text{if } m \text{ is even and if } n \text{ is odd; } \\ \dim H^0(L, \mathcal{I}_{Z'}(1,2)) & \text{otherwise,} \end{cases}$$

from which the proposition follows. \[ \square \]

To prove that $T(m, n; 1, 2; s)$ is true, it is therefore enough to prove that $R(m, n)$ is true if $m \leq n$. The proof is by double induction on $m$ and $n$. To be more precise, we first prove $R(m, m)$ and $R(m, m + 1)$ are true, and then we show that if $R(m, n - 2)$ is true, then $R(m, n)$ is also true.

Proposition 3.4. $R(m, m)$ is true for any $m \geq 1$.

Proof. Without loss of generality, we may assume that $U_L = \langle f_2, \ldots, f_{m+1} \rangle$. Let $p_0, \ldots, p_m \in \mathbb{P}^{m,m} \setminus L$. For each $i \in 0, \ldots, m$, we can consider $p_i$ as a pair $(u_i, v_i)$ of a vector $u_i$ in $V$ and $v_i$ in $W \setminus U_L$. Recall that

$$T_{p_i}(X_{m,m}) = V \otimes v^2 + u \otimes v_i W.$$ 
To prove the proposition, we will find explicit vectors $u_i$’s and $v_i$’s such that

$$V \otimes S_2(V) \equiv \sum_{i=0}^m T_{p_i}(X_{m,m}) \mod V \otimes S_2(U_L)).$$

Let $u_i = e_i$ for each $i \in \{0, \ldots, m\}$ and let

$$v_i = \begin{cases} f & \text{for } i = 0; \\ i f_0 + f_1 + f_i & \text{for } 2 \leq i \leq m. \end{cases}$$

Then we have

$$T_{p_i}(X_{m,m}) = \begin{cases} \langle e \otimes f^2_0, \ldots, \otimes f^2, e \otimes f f_1, \ldots, e \otimes f_0 f_m \rangle & \text{if } i = 0; \\ \langle e \otimes f^2_0, \ldots, \otimes f^2, e \otimes f f_1, \ldots, e \otimes f f_m \rangle & \text{if } i = 1; \\ \langle e \otimes (i f_0 + f_1 + f^2), \ldots, e \otimes (i f_0 + f_1 + f^2) \rangle & \text{if } i \geq 2. \end{cases}$$

Now we prove that every monomial in \{ $e_i \otimes f_j f_k \mid 0 \leq i, j, k \leq m$ \} lies in $\langle V \otimes S_2(U_L), \sum_{i=0}^m T_{p_i}(X_{m,m}) \rangle$.

For each $i \in \{2, \ldots, m\}$, we have

$$e \otimes (i f_0 + f_1 + f)^2 \equiv e \otimes (i^2 f^2_0 + f^2_0 + f^2 + 2 i f_0 f_1 + 2 i f_0 f_1 + 2 f_1 f_1)$$
$$\equiv e \otimes 2 f_1 f_1 \mod \langle V \otimes S_2(U_L), T_{p_1}(X_{m,m}), T_{p_2}(X_{m,m}) \rangle.$$
Indeed, $e_0 \otimes f_0^2, e_0 \otimes f_0f_1$ and $e_0 \otimes f_0f_1$ are in $T_{p_1}(X_{m,m}); e_0 \otimes f_1^2$ is in $T_{p_2}(X_{m,m}); e_0 \otimes f_1^2$ is in $V \otimes S_2(U_L).$ Similarly, one can prove that
\[ e_1 \otimes (i_0 + f_1 + f_1)^2 \equiv e_1 \otimes 2i_0 f_1 \text{ (mod } \langle V \otimes S_2(U_L), T_{p_1}(X_{m,m}), T_{p_2}(X_{m,m}) \rangle) \]
for each $i \in \{2, \ldots, m\}$. So we have proved that $e_1 \otimes f_j f_k \in \sum_{i=0}^{m} T_{p_i}(X_{m,m})$ if $i, j \in \{0, 1\}$ and $k \in \{0, \ldots, m\}$.

Note that, for each $i\neq j$ and $j \geq 2$, we have
\[ e_i \otimes (i_0 + f_1 + f_1)j \equiv e_i \otimes i_0 f_j + e_i \otimes f_1 f_j; \]
\[ e_i \otimes (j_0 + f_1 + f_1)^2 \equiv 2j e_i \otimes f_0 f_j + 2e_i \otimes f_1 f_j \]
modulo $\langle V \otimes S_2(U_L), \sum_{i=0}^{m} T_{p_i}(X_{m,m}) \rangle$. Hence
\[ e_i \otimes (j_0 + f_1 + f_1)^2 - 2j e_i \otimes (i_0 + f_1 + f_1)j \equiv \frac{2(i-j)}{i} e_i \otimes f_1 f_j. \]
This implies that $e_i \otimes f_1 f_j$, and hence $e_i \otimes f_0 f_j$, is contained in $\langle V \otimes S_2(U_L), \sum_{i=0}^{m} T_{p_i}(X_{m,m}) \rangle$, which completes the proof.

**Proposition 3.5.** $R(m, m+1)$ is true for any $m \geq 1$.

**Proof.** We prove that the statement is true if $m$ is even, because the other case can be proved in the same way.

Since $m$ is even, $s = \frac{3}{2} m + 1$. Let $p_1, \ldots, p_{\frac{m}{2}} \in L$ and let $q_1, \ldots, q_{m+1} \in \mathbb{P}^{m} \setminus L$. Choose an $\mathbb{P}^{m,m} = \mathbb{P}(V) \times \mathbb{P}(W) \subset \mathbb{P}^{m,m+1}$ in such a way that the intersection of $\mathbb{P}^{m,m}$ with $L$ is $\mathbb{P}^{m,m-2}$. We denote it by $H$. Specialize $q_1, \ldots, q_{m+1}$ in $H \setminus L$. Suppose that $p_1, \ldots, p_{\frac{m}{2}} \notin H$. Let $Z = \{p_1^2, \ldots, p_{\frac{m}{2}}^2, q_1^2, \ldots, q_{m+1}^2\}$. Then we have an exact sequence
\[ 0 \rightarrow I_{Z \cup L \cup H}(1, 2) \rightarrow I_{Z \cup L}(1, 2) \rightarrow I_{(Z \cup L) \cap H,H}(1, 2) \rightarrow 0. \]
By the assumption that $R(m, m)$ is true, dim $H^0(I_{(Z \cup L) \cap H,H}(1, 2)) = 0$. So we have
\[ \dim H^0(\mathbb{P}^{m,m+1}, I_{Z \cup L \cup H}(1, 2)) = \dim H^0(\mathbb{P}^{m,m+1}, I_{Z \cup L \cup H}(1, 2)). \]
Thus we need to prove that dim $H^0(\mathbb{P}^{m,m+1}, I_{Z \cup L \cup H}(1, 2)) = 1$.

Let $\tilde{Z}$ be the residual of $Z \cup L$ by $H$. Then $H^0(\mathbb{P}^{m,m+1}, I_{Z \cup L \cup H}(1, 2)) \approx H^0(\mathbb{P}^{m,m+1}, I_{\tilde{Z}}(1, 1))$. Note that $\tilde{Z}$ consists of $\frac{m}{2}$ double points $p_1^2, \ldots, p_{\frac{m}{2}}^2$, $m + 1$ single points $q_1, \ldots, q_{m+1}$ and $L$.

For simplicity, we also denote by $X_{m,m+1}$ the Segre variety obtained from $\mathbb{P}^{m,m+1}$ by embedding by the morphism given by $O(1, 1)$. The condition that dim $H^0(\mathbb{P}^{m,m+1}, I_{\tilde{Z}}(1, 1)) = 1$, i.e., $h_{p_{m,m+1}}(\tilde{Z}, (1, 1)) = (m+1)(m+2) - 1$, is equivalent to the condition that the following subspace of $V \otimes W$ has dimension $(m+1)(m+2) - 1$:
\[ \left\langle V \otimes U_L, \sum_{i=1}^{m/2} T_{p_i}(X_{m,m+1}), \sum_{i=1}^{m+1} \langle u'_i \otimes v'_i \rangle \right\rangle, \]
where $T_{p_i}(X_{m,m+1}) = V \otimes v_i + u_i \otimes W$ if $p_i = [u_i \otimes v_i] \in X_{m,m+1}$ and where $q_i = [u'_i \otimes v'_i]$. We may assume that $U_L = \langle f_0, \ldots, f_{m-1} \rangle$ and that $W = \langle f_1, \ldots, f_{m+1} \rangle$. Then

$$T_{p_i}(X_{m,m+1}) \equiv u_i \otimes \langle f_m, f_{m+1} \rangle \pmod{V \otimes U_L},$$

which implies that

$$\langle V \otimes U_L, T_{p_i}(X_{m,m+1}) \rangle = (V \otimes f_0) \oplus \left( V \otimes (U_L \cap W), \sum_{i=1}^{m/2} u_i \otimes \langle f_m, f_{m+1} \rangle \right).$$

Thus

$$\left\langle V \otimes U_L, \sum_{i=1}^{m/2} T_{p_i}(X_{m,m+1}), \sum_{i=1}^{m+1} \langle u'_i \otimes v'_i \rangle \right\rangle = (V \otimes f_0) \oplus \left( V \otimes (U_L \cap W'), \sum_{i=1}^{m/2} u_i \otimes \langle f_m, f_{m+1} \rangle, \sum_{i=1}^{m+1} (u'_i \otimes v'_i) \right).$$

Note that $T_1 = \{ e_i \otimes f_0 \mid 0 \leq i \leq m + 1 \}$ and $T_2 = \{ e_i \otimes f_j \mid 0 \leq i \leq m, 1 \leq j \leq m - 1 \}$ are bases for $V \otimes f_0$ and $V \otimes (U_L \cap W')$ respectively. Let $u_i = e_{i-1}$ for every $i \in \{1, \ldots, m\}$. Then $T_3 = \{ e_i \otimes f_j \mid 0 \leq i \leq m, 1 \leq j \leq m + 1 \}$. Let $T_4$ be the standard basis for $V \otimes W$ setminus $T_1 \cup T_2 \cup T_3$. Then $T_4$ consists of $m + 1$ distinct non-zero vectors. Choose $m + 1$ distinct elements of $T_4$ as $u'_i \otimes v'_i$s. Then $\bigcup_{i=1}^{d} T_i$ spans an $(m + 1)(m + 2) - 1$-dimensional vector space, which completes the proof. □

Let $U_M$ be another two-codimensional subspaces of $W$ and let $M$ be the subvariety of $\mathbb{P}^{m,n}$ of the forms $\mathbb{P}(V) \times \mathbb{P}(U_M)$. If $L$ and $M$ are general, then we have

$$\dim H^0(\mathbb{P}^{m,n}, I_{L \cup M}(1, 2)) = (m + 1) \left[ \binom{n + 2}{2} - 2 \binom{n}{2} + \binom{n - 2}{2} \right] = 4(m + 1).$$

This is equivalent to the condition that the subspace of $V \otimes W$ spanned by $V \otimes U_L$ and $V \otimes U_M$ has codimension $4(m + 1)$.

**Definition 3.6.** Let $p_1, \ldots, p_{m+1}$ be general points of $L$ and let $q_1, \ldots, q_{m+1}$ be general points of $M$. We denote by $W_{m,n}$ the subspace of $V \otimes S_2(W')$ spanned by $V \otimes S_2(U_L)$, $V \otimes S_2(U_M)$, $\sum_{i=1}^{m} T_{p_i}(X_{m,n})$ and $\sum_{i=1}^{m} T_{q_i}(X_{m,n})$. Then $\dim W_{m,n}$ is expected to be $(m + 1)(n^2 + 2)$. We say that $Q(m, n)$ is **true** if $W_{m,n}$ has the expected dimension.

**Remark 3.7.** Keeping the same notation as in the previous definition, we denote by $Z$ the zero-dimensional subscheme \{ $p_1^2, \ldots, p_{m+1}^2, q_1^2, \ldots, q_{m+1}^2$ \}. Then $Q(m, n)$ is true if and only if

$$\dim H^0(\mathbb{P}^{m,n}, I_{Z \cup L \cup M}(1, 2)) = 0.$$

**Proposition 3.8.** If $Q(m, n)$ and $R(m, n - 2)$ are true, then $R(m, n)$ is also true.

**Proof.** Let $p_1, \ldots, p_{s-(m+1)} \in L$ and let $q_1, \ldots, q_{m+1} \in \mathbb{P}^{m,n} \setminus L$. Suppose that $p_1, \ldots, p_{s-2(m+1)} \in L \cap M$, $p_{s-2m+1}, \ldots, p_{s-(m+1)} \in L \setminus L \cap M$ and $q_1, \ldots, q_{m+1} \in M$. Let $Z' = Z \cap M$. Then we have an exact sequence

$$0 \to I_{Z \cup L \cup M}(1, 2) \to I_{Z \cup L}(1, 2) \to I_{Z' \cup (L \cap M), M}(1, 2) \to 0.$$ 

Taking cohomology gives rise to the following exact sequence:

$$0 \to H^0(\mathbb{P}^{m,n}, I_{Z \cup L \cup M}(1, 2)) \to H^0(\mathbb{P}^{m,n}, I_{Z \cup L}(1, 2)) \to H^0(M, I_{Z' \cup (L \cap M), M}(1, 2)).$$

By the assumption $Q(m, n)$, $\dim H^0(\mathbb{P}^{m,n}, I_{Z \cup L \cup M}(1, 2)) = 0$. Thus we have

$$\dim H^0(\mathbb{P}^{m,n}, I_{Z \cup L}(1, 2)) \leq \dim H^0(M, I_{Z' \cup (L \cap M), M}(1, 2)).$$

Hence if $R(m, n - 2)$ is true, then so is $R(m, n)$. □

**Lemma 3.9.** If $Q(m - 2, n)$ and $Q(1, n)$ are true, then $Q(m, n)$ is also true.
Proof. Let $V'$ be a $(m - 1)$-dimensional subspace of $V$ and let $V''$ be a two-dimensional subspace of $V$. Suppose that $V$ can be written as the direct sum of $V'$ and $V''$. Let $U = \langle V \otimes S_2(U_L), V \otimes S_2(U_M) \rangle$. Suppose that if $p = ([u], [v^2]) \in \mathbb{P}^{m-2,n} = \mathbb{P}(V') \times \mathbb{P}(U_L)$. Then $V \otimes v^2 \subset V \otimes S_2(W)$. Thus

$$T_p(X_{m,n}) \equiv T_p(X_{m-2,n}) \mod V \otimes S_2(U_L).$$

Similarly, it can be proved that

$$T_q(X_{m,n}) \equiv T_q(X_{1,n}) \mod V \otimes S_2(U_L)$$

if $q = ([u], [v]) \in \mathbb{P}(V'^* \times \mathbb{P}(W)).$

This means that if $p_1, \ldots, p_{m+1} \in \mathbb{P}(V') \times \mathbb{P}(W)$ and if $q_1, \ldots, q_{m+1} \in \mathbb{P}(V'^*) \times \mathbb{P}(W)$, then

$$\sum_{i=1}^{m+1} T_{p_i}(X_{m,n}) \equiv \sum_{i=1}^{m+1} T_{q_i}(X_{m,n}) \mod V \otimes S_2(U_L).$$

In another word, $W_{m,n} \simeq W_{m-2,n} \oplus W_{1,n}$. Thus if $Q(m-2,n)$ and $Q(1,n)$ are true, so is $Q(m,n)$. □

Lemma 3.10. Let $n \geq 3$. Then $Q(1,n)$ and $Q(2,n)$ are true.

Proof. Here we only prove that $Q(1,n)$ is true for any $n \geq 3$, because the remaining case follows the same path.

To prove that $Q(1,n)$ is true, it is enough to prove that $Q(1,3)$ is true, because

$$\dim H^0(\mathbb{P}^{1,3}, \mathcal{I}_{Z \cup L \cup M}(1,2)) \geq \dim H^0(\mathbb{P}^{1,n}, \mathcal{I}_{Z \cup L \cup M}(1,2)).$$

Let $p_1$ and $p_2$ be general points of $L$ and let $q_1$ and $q_2$ be general points of $M$. To do so, we directly prove that

$$W_{1,3} = \langle V \otimes S_2(U_L), V \otimes S_2(U_M), T_{p_1}(X_{1,3}), T_{p_2}(X_{1,3}), T_{q_1}(X_{1,3}), T_{q_2}(X_{1,3}) \rangle.$$

Recall that $T_p(X_{1,3})$ for $p = [u \otimes v^2]$ is isomorphic to $V \otimes v^2 + u \otimes vW$. Thus one can check equality (3.1) as follows:

```plaintext
KK = ZZ/32003
S = KK[e_0..e_1];
R = KK[f_0..f_3];
SxR = KK[e_0..e_1,f_0..f_3];
v = sub(vars S,SxR);
bs = sub(basis(1,R),SxR);
p1 = v*random(source v,SxR{1:-1});
p2 = v*random(source v,SxR{1:-1});
q1 = v*random(source v,SxR{1:-1});
q2 = v*random(source v,SxR{1:-1});
w1 = ideal matrix{{f_0..1}};
p1' = gens w1*random(source gens w1,SxR{1:-1});
p2' = gens w1*random(source gens w1,SxR{1:-1});
w2 = ideal matrix{{f_2..3}};
q1' = gens w2*random(source gens w2,SxR{1:-1});
q2' = gens w2*random(source gens w2,SxR{1:-1});
t1 = ideal(p1*p1''+ideal(v**gens ideal(p1''-2)));
t2 = ideal(p2*p2''+ideal(v**gens ideal(p2''-2)));
t3 = ideal(q1*q1''+ideal(v**gens ideal(q1''-2)));
t4 = ideal(q2*q2''+ideal(v**gens ideal(q2''-2)));
L = v**gens (w1^2);
M = v**gens (w2^2);
bb = t1+t2+t3+t4+ideal L+ideal M;
```
betti trim bb

Since the ideal bb is minimally generated by twenty polynomials of multi-degree \(1, 2\), we completed the proof.

Theorem 3.11. Let \(n \geq 3\). Then \(Q(m, n)\) is true for any \(m\).

Proof. The proof is by two-step-induction on \(m\). Since we have already proved this proposition for \(m = 1\) and 2, we may assume that \(m \geq 3\). The statement \(Q(m, n)\) can be reduced to \(Q(m - 2, n)\) and \(Q(1, n)\). By induction hypothesis, \(Q(m - 2, n)\) is true. Since \(Q(1, n)\) is true by Lemma 3.10, it immediately follows from Proposition 3.9 that \(Q(m, n)\) is true.

As mentioned, we can prove the following claim as immediate consequences of Theorem 3.11:

Corollary 3.12. Let \(m \leq n\). Then \(R(m, n)\) is true.

Theorem 3.13. Suppose that \(m \leq n + 2\). Then \(T(m, n; 1, 2; s)\) is true for any \(s \leq s\).

Proof. Since \((m, n; 1, 2; s)\) is subabundant, it is enough to prove that \(T(m, n; 1, 2; s)\) is true. As claimed, the proof is by two-step-induction on \(m\) and \(n\). Clearly, \(T(m, m - 2; 1, 2; 0)\) is true. In Example 2.9, we proved that \(T(m, m - 1; 1, 2; s)\) is true.

Now suppose that \(T(m, n - 2; 1, 2; s - (m + 1))\) is true for some \(n\). We can reduce \(T(m, n; 1, 2; s)\) to \(T(m, n - 2; 1, 2; s - (m + 1))\) and \(R(m, n)\). By Corollary 3.12, \(R(m, n)\) is true. Thus it follows from Proposition 3.3 that \(T(m, n; 1, 2; s)\) is true.

Define a function \(r(m, n)\) in the following way:

\[
r(m, n) = \begin{cases} 
\frac{m^3 - 2m}{(m-2)(m+1)^2} & \text{if } m \text{ is even and if } n \text{ is odd;} \\
n & \text{otherwise.}
\end{cases}
\]

Corollary 3.14. Suppose that \(n > r(m, n)\). Then \(T(m, n; 1, 2; s)\) is true for any \(s \leq \left\lfloor \frac{(m+1)(n+2)}{m+n+1} \right\rfloor\).

Proof. Since \((m, n; 1, 2; s)\) is subabundant, it suffices to show that \(T(m, n; 1, 2; s)\) is true for \(s = \left\lfloor \frac{(m+1)(n+2)}{m+n+1} \right\rfloor\). Note that

\[
s = \begin{cases} 
(m + 1)k - \frac{(m-2)(m+1)}{2} + \frac{m^3-m}{2(m+n+1)} & \text{if } n \text{ is even;} \\
(m + 1)k - \frac{(m-3)(m+1)}{2} + \frac{m^3-m}{2(m+n+1)} & \text{if } m \text{ and } n \text{ are odd;} \\
(m + 1)k - \frac{(m-3)(m+1)+1}{2} + \frac{n+m^3+2}{2(m+n+1)} & \text{otherwise.}
\end{cases}
\]

It is straightforward to show that if \(n > r(m, n)\), then \(s = s\). Thus it follows immediately from Theorem 3.13 that \(T(m, n; 1, 2; s)\) is true.

Remark 3.15. If \(m = 1\), then \(r(1) < 0\). Since \(s = n + 1\), \(T(1, n; 1, 2; n+1)\) is true. Since \((1, n; 1, 2; n+1)\) is equiabundant, \(T(1, n; 1, 2; s)\) is therefore true for any \(s\).

4. Segre-Veronese varieties \(\mathbb{P}^m \times \mathbb{P}^n\) embedded by \(\mathcal{O}(1, 2)\): Superabundant Case

In this section, we keep the same notation as in Section 3. Let \(k = \left\lfloor \frac{n}{2} \right\rfloor\) and let

\[
\bar{s} = \begin{cases} 
(m + 1)k + 1 & \text{if } n \text{ is even;} \\
(m + 1)k + 3 & \text{otherwise.}
\end{cases}
\]

It is straightforward to show that \((m, n; 1, 2; \bar{s})\) is superabundant. The main goal of this section is to prove that \(T(m, n; 1, 2; \bar{s})\) is true, which implies that \(T(m, n; 1, 2; s)\) is true for any \(s \geq \bar{s}\).
Definition 4.1. Let $p_1, \ldots, p_{s-(m+1)}$ be general points of $L$, let $q_1, \ldots, q_{m+1}$ be general points of $\mathbb{P}^{m,n} \setminus L$ and let $V_{m,n}$ be the vector space $\left\langle V \otimes S^2(U_L), \sum_{i=1}^{\hat{s}-(m+1)} T_{p_i}(X_{m,n}), \sum_{i=1}^{m+1} T_{q_i}(X_{m,n}) \right\rangle$. Then the following inequality holds:

$$\dim V_{m,n} \leq (m+1) \left( \frac{n+2}{2} \right).$$

We say that $R(m,n)$ is true if the equality holds.

Remark 4.2. In the same way as in the proofs of Propositions 3.3 and 3.8, one can prove the following:

(i) If $R(m,n)$ and $T(m, n-2; 1, 2; \hat{s}-(m+1))$ are true, then $T(m, n; 1, 2; \hat{s})$ is true.

(ii) If $Q(m,n)$ and $R(m, n-2)$ are true, then $R(m, n)$ is true. In particular, if $R(m, n-2)$ is true, then $R(m, n)$ is true, because $Q(m, n)$ is true for $n \geq 3$.

Remark 4.2 (i) says that, to prove that $T(m,n; 1,2; s)$ is true for any $m$ and $n$ with $N \geq 1$, it is enough to show that $R(m,n)$ is true for such $m$ and $n$. Thus, by Remark 4.2 (ii), we only need to prove that $R(m,2)$ and $R(m,3)$ are true for any $m$.

Definition 4.3. Suppose that $(m,n) \neq (1,1)$. A 4-tuple $(m,n;1,d)$ is said to be balanced if

$$m \leq \left( \frac{n+d}{d} \right) - d.$$ 

Otherwise, we say that $(m,n;1,d)$ is unbalanced.

Remark 4.4. The notion of “unbalanced” was first introduced for Segre varieties (see for example [7, 2]). Then it was extended to Segre-Veronese varieties in [9]. They also proved that if $(m,n;1,d)$ is unbalanced, then $T(m,n; 1, d; s)$ fails if and only if

$$n < s < \min \left\{ m+1, \left( \frac{n+d}{d} \right) \right\}. \tag{4.1}$$

In particular, $T(m,2;1,2;m+1)$ is true if $m \geq 5$ and $T(m,3;1,2;m+1)$ is true if $m \geq 8$.

Here we would like to briefly explain that if $s$ satisfies the above inequalities, then $\sigma_s(X_{m,n})$ is defective. Let $p_1, \ldots, p_s$ be generic points on $X_{m,n}$. By assumption, we have $s < n+1$. Thus there is a proper subvariety of $\mathbb{P}^{m,n}$ of type $\mathbb{P}^{s-1,n}$ that contains $p_1, \ldots, p_s$. Thus we have

$$\dim \sigma_s(X_{m,n}) \leq s \left( \dim \mathbb{P}^{m,n} - \dim \mathbb{P}^{s-1,n} \right) + s \left( \frac{n+d}{d} \right)$$

$$= s \left[ \left( \frac{n+d}{d} \right) + m+1 - s \right].$$

It is straightforward to show that if $s$ fulfills Inequalities (4.1), then

$$s \left[ \left( \frac{n+d}{d} \right) + m+1 - s \right] < \min \left\{ s(m+n+1), (m+1) \left( \frac{n+d}{d} \right) \right\}.$$ 

Thus $\sigma_s(X_{m,n})$ is defective. This also says that, for such an $s$, the expected dimension of $\sigma_s(X_{m,n})$ is $s \left[ \left( \frac{n+d}{d} \right) + m+1 - s \right]$.

Lemma 4.5. Let $m \geq 2$.

(i) If $m \geq 3$, then $R(m,n)$ is true for any $n \geq 2$.

(ii) $R(2,n)$ is true for any $n \geq 3$.
Proof. We first prove the lemma for \( m \geq 8 \). As claimed, it suffices to show that \( \overline{R}(m, 2) \) and \( \overline{R}(m, 3) \) are true for any \( m \geq 8 \). We first prove the lemma for \( m \geq 8 \). Suppose that \( n \in \{2, 3\} \). If \( m \geq 8 \), then \((m, n; 1, 2)\) is unbalanced for each \( n \in \{2, 3\} \). Moreover, \((m, n; 1, 2; m + 1)\) is superabundant. Thus \( \overline{R}(m, n) \) can be reduced to \( T(m, n; 1, 2; m + 1) \). Since \( T(m, n; 1, 2; m + 1) \) is true, so is \( \overline{R}(m, n) \). Thus it remains to show that \( \overline{R}(m, n) \) is true for each \( m \in \{1, \ldots, m\} \).

This can be checked directly as follows:

```plaintext
codimensionTwo = (m,n)->(  
  k := floor(n/2);  
  K := ZZ/32003;  
  S := K[e_0..e_m];  
  R := K[f_0..f_n];  
  SxR := K[e_0..e_m,f_0..f_n];  
  v := sub(vars S,SxR);  
  bs := sub(vars R,SxR);  
  if n <=2 then w := ideal(0_(SxR)) else  
    w = ideal (f_0..f_(n-2));  
  if even n then a := 1;  
  if odd m and odd n then a = 2;  
  if even m and odd n then a = 3;  
  h := new MutableList;  
  scan((m+1)*k+a,i->h#i=v*random(source v,SxR^{1:-1}));  
  h = toList h;  
  h' := new MutableList;  
  scan((m+1)*(k-1)+a,i->h'#i=gens w*random(source gens w,SxR^{1:-1}));  
  h' = toList h';  
  h'' := new MutableList;  
  scan(m+1,i->h''#i=bs*random(source bs,SxR^{1:-1}));  
  h'' = toList h'';  
  h' = h'h'';  
  T := new MutableList;  
  scan((m+1)*k+a,i->T#i=ideal(h#i**h'#i**bs)+ideal(v**gens ideal((h'#i)^2))));  
  << (m+1)*binomial(n+2,2) << endl;  
  L := v**gens (w^2);  
  trim (ideal L + sum toList T)  
)
```

The command takes two integers \( m \) and \( n \) as input and returns \( \overline{V}_{m,n} \). Using this command, one can check that If \( m \geq 3 \) and if \( n = 2 \) or 3, then \( \overline{R}(m, n) \) is true. It can be also checked that \( \overline{R}(2, n) \) is true if \( n \geq 3 \). Thus we completed the proof.

Theorem 4.6. \( T(m, n; 1, 2; s) \) is true for any \( s \geq \bar{s} \).

Proof. Note that \( T(m, 1; 1, 2; s) \) is true for any \( s \). Since one can directly check that \( T(2, 2; 1, 2; 4) \) is true and since \( \overline{R}(2, n) \) is true for any \( n \geq 3 \), it follows from Remark 4.2 and Proposition 4.5 that \( T(2, n; 1, 2; \bar{s}) \) is true for any \( n \geq 1 \).

Suppose that \( m \geq 3 \). By Remark 4.2 and Proposition 4.5, it suffices to show that the theorem is true for \( n = 0 \) and 1. Suppose that \( n = 0 \). Then \( \bar{s} = 1 \). Clearly \( T(m, 0; 1, 2; 1) \) is true. Assume that \( n = 1 \). Then \( \bar{s} = 2 \) if \( m \) is odd; \( \bar{s} = 3 \) otherwise. As claimed, \( T(m, 1; 1, 2; s) \) is true for any \( s \). Thus we completed the proof.

5. Conjecture

Let \( X_{m,n} \) be the Segre-Veronese variety \( \mathbb{P}^{m,n} \) embedded by the morphism given by \( \mathcal{O}(1,2) \). The main purpose of this section is to give a conjecturally complete list of defective secant varieties of \( X_{m,n} \).
Let \( V \) be an \( m \)-dimensional vector space over \( \mathbb{C} \) with basis \( \{e_0, \ldots, e_m\} \) and let \( W \) be an \( n \)-dimensional vector space over \( \mathbb{C} \) with basis \( \{f_0, \ldots, f_n\} \). As mentioned at the beginning of Section 2, for a given point \( p = [u \otimes v^2] \in X_{m,n} \), the affine cone \( T_p(X_{m,n}) \) over the tangent space to \( X_{m,n} \) at \( p \) is isomorphic to \( V \otimes v^2 + u \otimes vW \) for a given point \( p = [u \otimes v^2] \in X_{m,n} \). Let \( A(p) \) be the \((m+1) \times (m+1)(\frac{n+2}{2})\) matrix whose \( i \)th row corresponds to \( e_i \otimes v^2 \) and let \( B(p) \) be the \((n+1) \times (m+1)(\frac{n+2}{2})\) matrix whose \( i \)th row corresponds to \( u \otimes vf_i \). Then \( T_p(X_{m,n}) \) is represented by the \((m+n+2) \times (m+1)(\frac{n+2}{2})\) matrix \( C(p) \) obtained by stacking \( A(p) \) and \( B(p) \):

\[
C(p) = ( A(p) \parallel B(p) ).
\]

For randomly chosen \( p_1, \ldots, p_s \in X_{m,n} \), let \( T_s(X_{m,n}) = \sum_{i=1}^{s} T_{p_i}(X_{m,n}) \). Then \( T_s(X_{m,n}) \) is represented by the \((m+n+2) \times (m+1)(\frac{n+2}{2})\) matrix \( C(p_1, \ldots, p_s) \) defined by

\[
C(p_1, \ldots, p_s) = ( C(p_1) \parallel C(p_2) \parallel \cdots \parallel C(p_s) ).
\]

Thus Remark 2.2 plus semi-continuity implies that if

\[
\text{rank } C(p_1, \ldots, p_s) = \min \left\{ s(m+n+1), (m+1)\left(\frac{n+2}{2}\right) \right\},
\]

then \( \sigma_s(X_{m,n}) \) has the expected dimension. If the above equality does not hold, this means that \( \sigma_s(X_{m,n}) \) is most likely defective. But it cannot be used to prove its deficiency, because one cannot pick true generic points. We programed this in Macaulay2 and computed the dimension of \( \sigma_s(X_{m,n}) \) for \( m, n \leq 20 \) to detect “potential” defective secant varieties of \( X_{m,n} \). This experiment shows that \( X_{m,n} \) is non-defective except for

- \((m,n;1,2)\) unbalanced;
- \((m,n) = (2,n)\), where \( n \) is odd and \( n \leq 20 \);
- \((m,n) = (4,3)\).

It is natural to guess that \( T(2,n;1,2;s) \) fails if \((n,s) = (2k+1,3k+2) \) with \( k \geq 1 \). Thus the experiment with our program suggests the following:

**Conjecture 5.1.** Let \( X_{m,n} \) be the Segre-Veronese variety \( \mathbb{P}^{m,n} \) embedded by the morphism given by \( \mathcal{O}(1,1) \). Then \( \sigma_s(X_{m,n}) \) is defective if and only if \((m,n,s)\) falls into one of the following cases:

- \((m,n;1,2)\) unbalanced and \((\frac{n+2}{2}) - n < \min \{m+1, \frac{n+2}{2}\}\);
- \((m,n,s) = (2,2k+1,3k+2) \) with \( k \geq 1 \);
- \((m,n) = (4,3,6)\).

Carlini and Chipalkatti have already shown that \( T(m,n;1,2;s) \) fails if \((m,n,s) = (2,3,5), (2,5,8) \) or \((4,3,6) \) in [8]. But, to our best knowledge, it is not yet proved that \( X_{2,n} \) is defective if \( n = 2k+1 \) with \( k \geq 3 \). To solve the conjecture, we therefore need to answer the following question:

**Question 5.2.** Is \( \sigma_{3k+2}(X_{2,2k+1}) \) defective for any \( k \geq 3 \)?

In [8], Carlini and Chipalkatti showed that the conjecture is true for \( m = 1 \). As an immediate consequence of Theorems 3.13 and 4.6, we prove that \( T(2,n;1,2;s) \) is true for any \( s \) except \((n,s) = (2k+1,3k+2) \) with \( k \geq 1 \).

**Theorem 5.3.** \( T(2,n;1,2;s) \) is true for any \( s \) except \((n,s) = (2k+1,3k+2) \) with \( k \geq 1 \).

**Proof.** Assume first that \( n = 2k \) is even. Then we have \( \overline{s} = 3k+1 \) and \( \underline{s} = 3k+1 \). Hence by Theorems 3.13 and 4.6, it follows that \( T(2,2k;1,2;s) \) is true for any \( s \).

Suppose now that \( n = 2k+1 \) is odd. Then we have \( \underline{s} = 3k+1 \) and \( \overline{s} = 3k+3 \). Thus \( T(2,n;1,2;s) \) is true for any \( s \leq 3k+1 \), by Theorem 3.13, and for any \( s \geq 3k+3 \), by Theorem 4.6.

If \( n = 1 \), then \( \underline{s} = 1 \) and \( \overline{s} = 3 \). So it remains only to prove that also \( T(2,1;1,2;2) \) is true. But this is proved in Example 2.9. \( \square \)
Lemma 5.5. We say that $R$ is true, it suffices to show that $\dim Z \geq 3^{(n+2)/2} - 2$, which means that if $\sigma_s(X_{m,n})$ is defective (as we expect), then it should be a hypersurface.

**Definition 5.4.** Suppose that $n$ is odd. Let $s = 3 \left\lfloor \frac{n}{2} \right\rfloor + 2$, let $p_1, \ldots, p_{s-3}$ be general points of $L$, let $q_1, \ldots, q_3$ be general points of $\mathbb{P}^{2,n} \setminus L$ and let $V_{2,n}$ be the vector space $\langle V \otimes S_2(U_L), \sum_{i=1}^{3} T_{p_i}(X_{m,n}), \sum_{i=1}^{3} T_{q_i}(X_{m,n}) \rangle$. Then the following inequality holds:

$$\dim V_{3,n} \leq 3 \left( \frac{n+2}{2} \right).$$

We say that $R(m,n)$ is true if the equality holds.

**Lemma 5.5.** Let $n$ be a positive odd integer greater than or equal to 3. Then $R(2,n)$ is true.

**Proof.** The proof is very similar to that of Proposition 3.5. One can easily prove that if $Q(2,n)$ is true and if $R(2,n-2)$ is true, then $R(2,n)$ is true. Since we have already proved that $Q(2,n)$ is true, it suffices to show that $R(2,3)$ is true.

Let $p_1, p_2 \in L$ and let $q_1, q_2, q_3 \in \mathbb{P}^{2,n}$. Choose a subvariety $H$ of $\mathbb{P}^{2,3}$ of the form $\mathbb{P}^{2,2} = \mathbb{P}(V) \times \mathbb{P}(W)$ such that $\mathbb{P}^{2,2}$ intersects $L$ in $\mathbb{P}^{2,0}$. Suppose that $p_1, p_2 \not\in H$. Specializing $q_1, q_2$ and $q_3$ in $H \setminus L$, we obtain an exact sequence:

$$0 \to I_{Z \cup L \setminus H}(1,2) \to I_{Z \cup L}(1,2) \to I_{(Z \cup L) \cap H}(1,2) \to 0,$$

where $Z = \{p_1^2, p_2^2, q_1^2, q_2^2\}$. Since we have already proved that $R(2,2)$ is true, we can conclude that $\dim H^0(I_{(Z \cup L) \cap H}(1,2)) = 0$. Thus it is enough to prove that $H^0(I_{Z \cup L \setminus H}(1,2)) = 0$ or $H^0(I_{\bar{Z}}(1,1)) = 0$, where $\bar{Z}$ is the residual of $Z \cup L$ by $H$. Note that $\bar{Z}$ consists of two double points $p_1^2, p_2^2$, three single points $q_1, q_2, q_3$ and $L$. Let $X_{2,3}$ be the Segre-Veronese variety $\mathbb{P}^{2,3}$ embedded by $O(1,1)$. We want to prove that $L, \sum_{i=1}^{2} T_{p_i}(X_{2,3})$ and $\sum_{i=1}^{3} T_{q_i}(X_{2,3})$ span $V \otimes W$. Note that if $p = u \otimes v$, then $T_p(X_{2,3}) = V \otimes v + u \otimes W$. Now assume the following:

- $U_L = \langle f_0, f_1 \rangle$ and $W' = \langle f_1, f_2, f_3 \rangle$;
- $p_1 = e_0 \otimes f_0, p_2 = e_1 \otimes f_1 \in V \otimes U_L$;
- $q_1 = e_2 \otimes f_2, q_2 = e_3 \otimes f_3 \in V \otimes W'$.

For any non-zero $q_3 \in V \otimes W'$, one can show that

$$V \otimes W = \langle L, \sum_{i=1}^{2} T_{p_i}(X_{2,3}), \sum_{i=1}^{3} T_{q_i}(X_{2,3}) \rangle.$$ 

Thus we completed the proof. □

**Proposition 5.6.** If $(n,s) = (2k+1,3k+2)$ for $k \geq 1$, then $\dim \sigma_s(X_{2,3}) \geq 3^{(n+2)/2} - 2$.

**Proof.** The proof is by induction on $k$. In [8], it has been already proved that $\sigma_5(X_{2,3})$ is a hypersurface. Thus we may assume that $k \geq 2$. Let $p_1, \ldots, p_s \in \mathbb{P}^{2,n}$. Then there is a subvariety $L$ of $\mathbb{P}^{2,n}$ of the form $\mathbb{P}^{2,n-2}$ such that $p_1, p_2, p_3 \in L$. Let us suppose that $p_2, \ldots, p_s \in \mathbb{P}^{2,n} \setminus L$. Then we have an exact sequence

$$0 \to I_{Z \cup L}(1,2) \to I_{Z}(1,2) \to I_{Z \cap L}(1,2) \to 0.$$ 

Taking cohomology, we get

$$\dim H^0(I_{Z}(1,2)) \leq \dim H^0(I_{Z \cup L}(1,2)) + \dim H^0(I_{Z \cap L}(1,2)).$$

By Lemma 5.5, $\dim H^0(I_{Z \cup L}(1,2)) = 0$. Thus, by induction hypothesis,

$$\dim H^0(I_{Z}(1,2)) \leq \dim H^0(I_{Z \cap L}(1,2)) \leq 1,$$

which completes the proof. □
Remark 5.7. As claimed, it is known that $T(2,5;1,2;8)$ does not hold. Hence Proposition 5.6 implies that $\sigma_8(X_{2,5})$ is a hypersurface in $\mathbb{P}^{62}$.

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