ON NON-DEFECTIVITY OF CERTAIN SEGRE-VERONESE VARIETIES

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ABSTRACT. Let $X_{m,n}$ be the Segre-Veronese variety $\mathbb{P}^m \times \mathbb{P}^n$ embedded by the morphism given by $\mathcal{O}(1,2)$ and let $\sigma_s(X_{m,n})$ denote the s^{th} secant variety to $X_{m,n}$. In this paper, we prove that if $m = n$ or $m = n+1$, then $\sigma_s(X_{m,n})$ has the expected dimension except for $\sigma_6(X_{4,3})$. As an immediate consequence, we will give function $s_1(m,n) \leq s_2(m,n)$ such that if $s \leq s_1(m,n)$ or if $s \geq s_2(m,n)$, then $\sigma_s(X_{m,n})$ has the expected dimension.

1. INTRODUCTION

Let $X \subseteq \mathbb{P}^{N-1}$ be a non-degenerate projective variety of dimension d. Then the s^{th} *secant variety* of X, denoted $\sigma_s(X)$, is the Zariski closure of the union of linear spans of s-tuples of points lying on X. The major questions surrounding secant varieties center around finding invariants of those objects such as dimension. A simple dimension count suggests that the expected dimension of $\sigma_s(X)$ is $\min\{N-1, s(d+1)-1\}$. We say that X has a *defective* s^{th} *secant variety* if $\sigma_s(X)$ does not have the expected dimension. In particular, X is said to be *defective* if X has a defective s^{th} secant variety for some s.

The paper explores problems related to the classification of defective secant varieties of Sege-Veronese varieties. This is analogous to the celebrated theorem of Alexander and Hirschowitz [5], which asserts that higher secant varieties of Veronese varieties have the expected dimension (modulo a fully described list of exceptions). This work completed the *Waring problem* for polynomials which had remained unsolved for some time. There are corresponding conjecturally complete lists of defective secant varieties for Segre varieties [3] and for Grassmann varieties [7]. Secant varieties of Segre-Veronese varieties are however less well-understood. In recent years, considerable efforts have been made to develop techniques to study secant varieties of Segre-Veronese varieties (see for example [11, 10, 9, 15, 13, 6, 12, 4]). But even the classification of defective two-factor Segre-Veronese varieties is still far from complete.

A useful tool to study secant varieties to a Segre-Veronese variety is a technique based on specializing points to lie on a certain divisor of the Segre-Veronese variety. This technique allows one to compute the dimension of the secant variety by induction on dimension and degree. The secant varieties of Segre-Veronese varieties $X_{m,n}$ obtained by embedding $\mathbb{P}^m \times \mathbb{P}^n$ into $\mathbb{P}^{(m+1)\binom{n+2}{2}-1}$ by the morphism given by $\mathcal{O}(1,2)$ can be viewed as an initial case of this induction approach, Indeed, the above-mentioned specialization technique involves secant varieties of two-factor

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Segre varieties, most of which are known to be defective. Thus one needs a different approach to classify defective secant varieties of such Segre-Veronese varieties.

Very recently, secant varieties of Segre-Veronese varieties $X_{m,n}$ were systematically studied in [1], where the authors suggested that secant varieties of such Segre-Veronese varieties are not defective modulo the list of the well known exceptions. More precisely, they proposed the following conjecture:

Conjecture 1.1 ([1]). *Let $X_{m,n}$ be the Segre-Veronese variety obtained from $\mathbb{P}^m \times \mathbb{P}^n$ by embedding in $\mathbb{P}^{(m+1)\binom{n+2}{2}-1}$ by the morphism given by $\mathcal{O}(1,2)$. Then $\sigma_s(X_{m,n})$ is defective if and only if (m,n,s) falls into one of the following cases:*

(i) *$(m,n;1,2)$ is unbalanced, i.e., $m > \binom{n+2}{2} - 2$, and*

$$\binom{n+2}{2} - n < s < \min\left\{m+1,\ \binom{n+2}{2}\right\};$$

(ii) *$(m,n,s) = (2, 2k+1, 3k+2)$ with $k \geq 1$;*

(iii) *$(m,n,s) = (4,3,6)$.*

The existence of the listed defective cases were established by several authors [10, 15]. Evidence for Conjecture 1.1 was given by a mixture of theory and computational experiments. The main goal of this paper is to provide further evidence in support of this conjecture. As a first step, we will prove the following theorem:

Theorem 1.2. *Let $n \geq 1$.*

(i) *$X_{n,n}$ is not defective.*

(ii) *$X_{n,n-1}$ is not defective except for $n = 4$.*

To prove Theorem 1.2, we will utilize a technique based on specialization of points on certain subvarieties (but not on divisors). This specialization technique enables one to compute the dimensions of secant varieties by induction on dimension. A more precise description of this technique can be found in Sections 3 and 4. Note that very similar approaches were successfully applied to study secant varieties of other well known varieties such as Veronese varieties and Grassmann varieties (see [8] and [2]).

In Section 5, we will develop a "double induction" approach that is a modification of the approach discussed in [1]. This induction approach together with Theorem 1.2 as an initial case allows us to establish the existence of a large family of non-defective secant varieties of $X_{m,n}$. To be more precise, we will prove the following theorem:

Theorem 1.3. *Suppose that $n \geq m-1$.*

(i) *Let $k = \lfloor n/2 \rfloor$ and let $s_1(m,n)$ be the function defined by*

$$s_1(m,n) = \begin{cases} (m+1)k - (m-2)(m+1)/2 & \text{if } n \text{ is even;} \\ (m+1)k - (m-3)(m+1)/2 & \text{if } m \text{ and } n \text{ are odd;} \\ (m+1)k - [(m-3)(m+1)+1]/2 & \text{otherwise.} \end{cases}$$

Then $\sigma_s(X_{m,n})$ has the expected dimension for any $s \leq s_1(m,n)$.

(ii) *Let $s_2(m,n)$ be the function defined by*

$$s_2(m,n) = \begin{cases} \overline{s}(m,m) + (m+1)(n-m)/2 & \text{if } 2|(n-m) \\ \overline{s}(m,m-1) + (m+1)(n-m+1)/2 & \text{if } 2 \not| (n-m), \end{cases}$$

where $\overline{s}(m,m) = \lceil (m+1)\binom{m+2}{2}/(2m+1) \rceil$ *and* $\overline{s}(m,m-1) = \lceil (m+1)\binom{m+1}{2}/2m \rceil$. *Then* $\sigma_s(X_{m,n})$ *has the expected dimension for any* $s \geq s_2(m,n)$.

Note that if $s(m+n+1) \leq (m+1)\binom{n+2}{2}$ and if $\sigma_s(X_{m,n})$ has the expected dimension, then $\sigma_{s'}(X_{m,n})$ also has the expected dimension for any $s' \leq s$. Likewise, if $s \geq (m+1)\binom{n+2}{2}$ and if $\sigma_s(X_{m,n})$ has the expected dimension, then $\sigma_{s'}(X_{m,n})$ also has the expected dimension for any $s' \geq s$. In particular, $X_{m,n}$ is non-defective if and only if $\sigma_s(X_{m,n})$ has the expected dimension for $s = \lfloor f(m,n) \rfloor$ and $\lceil f(m,n) \rceil$, where $f(m,n) = (m+n+1)^{-1}(m+1)\binom{n+2}{2}$. Thus, ideally, we would like $s_1(m,n)$ and $s_2(m,n)$ to satisfy $s_1(m,n) \leq s_2(m,n) - 1$. While Theorem 1.3 does not reach this result, the difference $s_2(m,n) - s_1(m,n)$ is relatively small. Indeed, Theorem 1.3 (ii) significantly improves the bound presented in [1].

The proof of Theorem 1.2 is partially based on computations in `Macaulay2`. (`Macaulay2`, developed by D. Grayson and M. Stillman [14], is a widely used computer algebra system for research in algebraic geometry and commutative algebra.) All the `Macaulay2` scripts needed to make these computations are available at `http://www.webpages.uidaho.edu/~abo/programs.html`.

2. Preliminaries

Let V be an $(m+1)$-dimensional vector space over $\mathbb{C}$ with basis $\{e_0, \ldots, e_m\}$ and let W be an $(n+1)$-dimensional vector space over $\mathbb{C}$ with basis $\{f_0, \ldots, f_n\}$. For simplicity, we write $Y_{m,n}$ for $\mathbb{P}^m \times \mathbb{P}^n = \mathbb{P}(V) \times \mathbb{P}(W)$. In this section, we indicate by $X_{m,n}$ for the Segre-Veronese variety $Y_{m,n}$ embedded by the morphism given by $\mathcal{O}(1,2)$. For simplicity, if $m = n$, then we denote $X_{m,n}$ by X_n and $Y_{m,n}$ by Y_n.

Let $T_p(X_{m,n})$ be the affine cone over the tangent space $\mathbb{T}_p(X_{m,n})$ to $X_{m,n}$ at a point $p \in X_{m,n}$. For each $p \in X_{m,n}$, there are $v \in V \setminus \{0\}$ and $w \in W \setminus \{0\}$ such that $p = [v \otimes w^2] \in \mathbb{P}(V \otimes S_2(W))$. In this case, $T_p(X_{m,n}) = V \otimes w^2 + v \otimes wW$.

Definition 2.1. Let $p_1, \ldots, p_s$ be general points of $X_{m,n}$. For simplicity, we write $E_{m,n,s}$ for $\sum_{i=1}^{s} T_{p_i}(X_{m,n})$. Then $E_{m,n,s}$ is expected to have dimension

$$e(m,n) = \min\{s(m+n+1), N(m,n)\},$$

where $N(m,n) = (m+1)\binom{n+2}{2}$. We say that *the statement* $T(m,n;s)$ *is true* if $E_{m,n,s}$ has the expected dimension. We also say that *the statement* $T(m,n)$ *is true* if $T(m,n;s)$ is true for every non-negative integer s. If $m = n$, we write $T(n;s)$ and $T(n)$ instead of $T(m,n;s)$ and $T(m,n)$ respectively.

Remark 2.2. Let $p_1, \ldots, p_s$ be general points of $X_{m,n}$ and let $\sigma_s(X_{m,n})$ be the s^{th} secant variety of $X_{m,n}$. By Terracini's lemma [16], the span of the tangent spaces to $X_{m,n}$ at s generic points is equal to the tangent space to $\sigma_s(X_{m,n})$ at the generic point z on the linear subspace spanned by the s points. Thus $E_{m,n,s}$ can be thought of as the affine cone over the tangent space to $\sigma_s(X_{m,n})$ at z. Therefore, $T(m,n;s)$ is true if and only if $E_{m,n,s}$ has the expected dimension.

For a given triple (m,n,s), we can check whether $T(m,n;s)$ is true as follows: Choose randomly $p_1, \ldots, p_s \in X_{m,n}$. Then one can compute $E_{m,n,s}$ explicitly. Let $\mathfrak{R} = \mathbb{C}[e_0, \ldots, e_m, f_0, \ldots, f_n]$ and let I be the ideal of $\mathfrak{R}$ generated by $E_{m,n,s}$. Minimize generators for I (this can be done by a computer algebra system such as `Macaulay2`). If $e(m,n)$ multi-homogeneous polynomials of degree $(1,2)$ form a minimal generating set of I, then $T(m,n;s)$ is true.

Remark 2.3. The cohomology group $H^0(Y_{m,n}, \mathcal{O}(1,2))$ can be identified with the set of hyperplanes in $\mathbb{P}^{N(m,n)-1}$. Since the condition that a hyperplane $H \subset \mathbb{P}^{N(m,n)-1}$ contains $\mathbb{T}_p(X_{m,n})$ is equivalent to the condition that H intersects $X_{m,n}$ in the first infinitesimal neighborhood p^2 of p, the elements of $H^0(Y_{m,n}, \mathcal{I}_{p^2}(1,2))$ can be viewed as hyperplanes containing $\mathbb{T}_p(X_{m,n})$.

Let $p_1, \ldots, p_s \in X_{m,n}$ and let $Z = \{p_1^2, \ldots, p_s^2\}$. Recall that Terracini's lemma says that the linear subspace spanned by $\mathbb{T}_{p_1}(X_{m,n}), \ldots, \mathbb{T}_{p_s}(X_{m,n})$ is the tangent space to $\sigma_s(X_{m,n})$ at a general point on the linear subspace spanned by $p_1, \ldots, p_s$. This implies that the dimension of $\sigma_s(X_{m,n})$ equals the value of the Hilbert function $h_{Y_{m,n}}(Z, \cdot)$ of Z at $(1,2)$, i.e.,

$$h_{Y_{m,n}}(Z, (1,2)) = \dim H^0(Y_{m,n}, \mathcal{O}(1,2)) - \dim H^0(Y_{m,n}, \mathcal{I}_Z(1,2)).$$

In particular,

$$h_{Y_{m,n}}(Z, (1,2)) = \min\{s(m+n+1),\ N(m,n)\}$$

if and only if $T(m,n;s)$ is true.

Remark 2.4. Let s be a positive integer. Suppose that $s(m+n+1) \leq N(m,n)$ and that $T(m,n;s)$ is true. Then $T(m,n;s')$ is true for any choice of s' with $s \geq s'$. Similarly, if $s(m+n+1) \geq N(m,n)$ and if $T(m,n;s)$ is true, then $T(m,n;s')$ is true for any choice of s' with $s \leq s'$. In order to prove that $T(m,n;s)$ is true, it is therefore enough to show that both $T(m,n;\underline{s}(m,n))$ and $T(m,n;\overline{s}(m,n))$ are true, where $\underline{s}(m,n) = \lfloor N(m,n)/(m+n+1) \rfloor$ and $\overline{s}(m,n) = \lceil N(m,n)/(m+n+1) \rceil$.

3. Segre varieties $\mathbb{P}^n \times \mathbb{P}^n$ embedded by $\mathcal{O}(1,2)$

Let V and W be $(n+1)$-dimensional vector spaces over $\mathbb{C}$ with bases $S = \{e_0, \ldots, e_n\}$ and $T = \{f_0, \ldots, f_n\}$ respectively. Let $N(n) = (n+1)\binom{n+2}{2}$. As in Section 2, we write Y_n for $\mathbb{P}^n \times \mathbb{P}^n = \mathbb{P}(V) \times \mathbb{P}(W)$ and X_n for the Segre-Veronese variety obtained by embedding Y_n into $\mathbb{P}^{N(n)-1}$ by the morphism given by $\mathcal{O}(1,2)$. This section will be devoted to proving that X_n is not defecitve for every positive integer n.

For each $i \in \{1,2,3\}$, let $S_i = \{e_{n-8i+1}, \ldots, e_{n-8(i-1)}\}$, let $T_i = \{f_{n-8i+1}, \ldots, f_{n-8(i-1)}\}$, let $V_i = \langle S \setminus S_i \rangle$ and let $W_i = \langle T \setminus T_i \rangle$. We denote by L_i the subvariety $\mathbb{P}(V_i) \times \mathbb{P}(W_i)$ of Y_n. Recall that if $p = [v \otimes w^2] \in X_n$, then the affine cone $T_p(X_n)$ over the tangent space to X_n at p is $V \otimes w^2 + v \otimes wW$. Note that if p lies in L_i, then $T_p(X_n)$ is congruent to $\langle S_i \rangle \otimes w^2 + v \otimes w\langle T_i \rangle$ modulo $V_i \otimes S_2(W_i)$ for each $i \in \{1,2,3\}$. Thus the dimension of the subspace of $V \otimes S_2(W)$ spanned by $T_p(X_n)$ and $V_i \otimes S_2(W_i)$ is less than or equal to $N(n-8)+16$.

Definition 3.1. Let $n \geq 24$. For each $i \in \{1,2,3\}$, let $p_{i,1}, \ldots, p_{i,32}$ be general points of L_i. We denote by A_n the following subspace of $V \otimes S_2(V)$:

$$\sum_{i=1}^{3} \left[V_i \otimes S_2(W_i) + \sum_{j=1}^{32} T_{p_{i,j}}(X_n) \right].$$

Note that

$$\dim \left[\sum_{i=1}^{3} V_i \otimes S_2(W_i) \right] = 3N(n-8) - 3N(n-16) + N(n-24).$$

Thus the dimension of A_n is expected to be

$$3N(n-8) - 3N(n-16) + N(n-24) + 3 \cdot 32 \cdot 16 = N(n).$$

We say that *the statement $R(n)$ is true* if A_n has the expected dimension.

Remark 3.2. For a fixed positive integer $n \geq 24$, we can check whether $R(n)$ is true as follows: Let $\mathfrak{R} = \mathbb{C}[e_0, \ldots, e_n, f_0, \ldots, f_n]$. For each $i \in \{1,2,3\}$, select $p_{i,1}, \ldots, p_{i,32} \in L_i$ at random. Then one can compute A_n explicitly. Let I be the ideal of $\mathfrak{R}$ generated by A_n. Minimize generators for I. If the minimal generating set of I consists of $N(n)$ multi-homogeneous polynomials of degree $(1,2)$, then $R(n)$ is true.

Remark 3.3. Since $\dim H^0(Y_n, \mathcal{O}(1,2)) = N(n)$, the statement $R(n)$ is true if and only if

$$\dim H^0(Y_n, \mathcal{I}_{L_1 \cup L_2 \cup L_3 \cup Z}(1,2)) = 0,$$

where $Z = \{p_{i,1}^2, \ldots, p_{i,32}^2\}_{1 \leq i \leq 3}$.

Proposition 3.4. *$R(n)$ is true for evey $n \geq 24$.*

Proof. For each $i \in \{1,2,3\}$, let $p_{i,1}, \ldots, p_{i,32}$ be general points of L_i and let $Z = \{p_{i,1}^2, \ldots, p_{i,32}^2\}_{1 \leq i \leq 3}$. For simplicity, we denote $L_1 \cup L_2 \cup L_3$ by L. By Remark 3.3, it suffices to show $H^0(Y_n, \mathcal{I}_{L \cup Z}(1,2)) = 0$.

Let V_K be the subspace of V spanned by S_1, S_2 and S_3, let W_K be the subpsace of W spanned by T_1, T_2 and T_3 and let $K = \mathbb{P}(V_K) \times \mathbb{P}(W_K)$. It is straightforward to show that $V \otimes S_2(W) = V_K \otimes S_2(W_K) + \sum_{i=1}^3 V_i \otimes S_2(W_i)$. In another word, $H^0(Y_n, \mathcal{I}_{L \cup K}(1,2)) = 0$. Let V_H be a general subspace of V of dimension greater than or equal to 25 that contains V_K, let W_H be a general subspace of W of dimension greater than or equal to 25 that contains W_K and let $H = \mathbb{P}(V_H) \otimes \mathbb{P}(W_H)$. Then $H^0(Y_n, \mathcal{I}_{L \cup H}(1,2)) = 0$, because $\dim H^0(Y_n, \mathcal{I}_{L \cup H}(1,2)) \leq \dim H^0(Y_n, \mathcal{I}_{L \cup K}(1,2))$.

Specializing $p_{i,1}, \ldots, p_{i,32}$ to H, we get the following short exact sequence:

$$0 \to \mathcal{I}_{L \cup Z \cup H}(1,2) \to \mathcal{I}_{L \cup Z}(1,2) \to \mathcal{I}_{(L \cup Z) \cap H, H}(1,2) \to 0.$$

Since

$$\dim H^0(Y_n, \mathcal{I}_{L \cup H}(1,2)) \geq \dim H^0(Y_n, \mathcal{I}_{L \cup Z \cup H}(1,2)),$$

we have $H^0(Y_n, \mathcal{I}_{L \cup Z \cup H}(1,2)) = 0$. So it is enough to prove the truth of $R(24)$. One can check that $R(24)$ holds in the same way as described in Remark 3.2. Thus we completed the proof. □

Definition 3.5. For each non-negative integer n, let k and r be the quotient and remainder when dividing n by 8. For every $n \geq 2$, we define a function $\underline{s}(n)$ by

$$\underline{s}(n) = \begin{cases} 16k^2 + 7k & \text{if } r = 0 \\ 16k^2 + 11k + 1 & \text{if } r = 1 \\ 16k^2 + 15k + 3 & \text{if } r = 2 \\ 16k^2 + 19k + 5 & \text{if } r = 3 \\ 16k^2 + 23k + 8 & \text{if } r = 4 \\ 16k^2 + 27k + 11 & \text{if } r = 5 \\ 16k^2 + 31k + 15 & \text{if } r = 6 \\ 16k^2 + 34k + 19 & \text{if } r = 7 \end{cases}$$

and a function $\overline{s}(n)$ by $\underline{s}(n) + 1$. Note that $\underline{s}(n) = \lfloor N(n)/(2n+1) \rfloor$ and $\overline{s}(n) = \lceil N(n)/(2n+1) \rceil$.

Let $t(n) = \underline{s}(n-8) - \underline{s}(n-16) = \overline{s}(n-8) - \overline{s}(n-16) = 32k + 4r - 41$. For each $i \in \{1, 2\}$, let $p_{i,1}, \ldots, p_{i,t(n)}$ be general points of L_i and let $p_1, \ldots, p_{32}$ be general points of Y_n. Suppose that $n \geq 16$. Let B_n be the following subspace of $V \otimes S_2(W)$:

$$\sum_{i=1}^{2} \left[V_i \otimes S_2(W_i) + \sum_{j=1}^{t(n)} T_{p_{i,j}}(X_n) \right] + \sum_{\ell=1}^{32} T_{p_\ell}(X_n).$$

A simple dimension count suggests that B_n is expected to have dimension

$$2N(n-8) - N(n-16) + 2 \cdot 16 \cdot t(n) + 32 \cdot (2n+1) = N(n).$$

We say that *the statement $Q(n)$ is true* if B_n has the expected dimension. This is equivalent to the condition that

$$\dim H^0(X_n, \mathcal{I}_{L_1 \cup L_2 \cup Z}(1,2)) = 0,$$

where $Z = \{p_{1,1}^2, \ldots, p_{1,t(n)}^2, p_{2,1}^2, \ldots, p_{2,t(n)}^2, p_1^2, \ldots, p_{32}^2\}$.

Remark 3.6. For a fixed positive integer $n \geq 16$, one can check whether or not $Q(n)$ is true in the same way as in Remark 3.2. Let $\mathfrak{R} = \mathbb{C}[e_0, \ldots, e_n, f_0, \ldots, f_n]$. Select $p_{i,1}, \ldots, p_{i,t(n)} \in L_i$ and $p_1, \ldots, p_{32} \in Y_n$ at random. Then compute B_n explicitly. Let I be the ideal of $\mathfrak{R}$ generated by B_n. Minimize generators for I. Then $Q(n)$ is true if the minimal generating set of I consists of $N(n)$ multi-homogeneous polynomials of degree $(1, 2)$.

Proposition 3.7. *$Q(n)$ is true for every $n \geq 16$.*

Proof. The proof is by eight-step induction on n. Let L_3 be the subvariety of Y_n defined at the beginning of this section. Note that $L_i \cap L_3$ is isomorphic to Y_{n-16} for each $i \in \{1, 2\}$. Specialize $p_{i,1}, \ldots, p_{i,t(n)-32}$ to $L_i \cap L_3$ and $p_1, \ldots, p_{32}$ to L_3. Consider the following exact sequence:

$$0 \to \mathcal{I}_{L_1 \cup L_2 \cup L_3 \cup Z}(1,2) \to \mathcal{I}_{L_1 \cup L_2 \cup Z}(1,2) \to \mathcal{I}_{(L_1 \cup L_2 \cup Z) \cap L_3, L_3}(1,2) \to 0.$$

It follows from Proposition 3.4 that $H^0(Y_n, \mathcal{I}_{L_1 \cup L_2 \cup L_3 \cup Z}(1,2)) = 0$. Thus it is sufficient to show that $H^0(Y_n, \mathcal{I}_{L_1 \cup L_2 \cup Z}(1,2)) = 0$ for every $n \in \{16, \ldots, 23\}$. This can be done in the same way as described in Remark 3.6. □

Definition 3.8. Keeping in mind the notation of Definition 3.5, let $p_1, \ldots, p_{t(n+8)}$ be general points of Y_n and let $q_1, \ldots, q_{\underline{s}(n-8)}$ be general points of L_1. We denote by $\underline{C}_n$ the following subspace of $V \otimes S_2(W)$:

$$\sum_{i=1}^{t(n+8)} T_{p_i}(X_n) + \sum_{i=1}^{\underline{s}(n-8)} T_{q_i}(X_n) + L_1. \tag{3.1}$$

Let $\overline{C}_n$ be the subspace obtained from (3.1) by replacing $\underline{s}(n-8)$ by $\overline{s}(n-8)$. The subspace $\underline{C}_n$ is expected to have dimension

$$c(n) = (2n+1) \cdot t(n+8) + 16 \cdot \underline{s}(n-8) + N(n-8),$$

while $\overline{C}_n$ is expected to coincide with $V \otimes S_2(W)$. We say that the statement $\underline{O}(n)$ (resp. $\overline{O}(n)$) *is true* if $\underline{C}_n$ (resp. $\overline{C}_n$) has the expected dimension.

Remark 3.9. For a fixed positive integer $n \geq 8$, let $\mathfrak{R} = \mathbb{C}[e_0, \ldots, e_n, f_0, \ldots, f_n]$. Select $p_1, \ldots, p_{t(n+8)} \in Y_n$ and $q_1, \ldots, q_{\underline{s}(n-8)} \in L_1$ at random. This allows one to describe $\underline{C}_n$ explicitly. Let I be the ideal of $\mathfrak{R}$ generated by $\underline{C}_n$. Minimize generators for I. The statement $\underline{O}(n)$ is true if the minimal generating set of

I consists of $c(n)$ multi-homogeneous polynomials of degree $(1,2)$. One can also check whether or not $\overline{O}(n)$ is true in a similar manner.

Remark 3.10. Let $Z = \{p_1^2, \ldots, p_{t(n+8)}^2, q_1^2, \ldots, q_{\underline{s}(n-8)}^2\}$. Then $\underline{O}(n)$ is true if and only if the following holds:

$$\dim H^0(Y_n, \mathcal{I}_{L_1 \cup Z}(1,2)) = \begin{cases} 13 & \text{if } n \equiv 0 \pmod 8 \\ 15 & \text{if } n \equiv 1 \pmod 8 \\ 9 & \text{if } n \equiv 2 \pmod 8 \\ 11 & \text{if } n \equiv 3 \pmod 8 \\ 5 & \text{if } n \equiv 4 \pmod 8 \\ 7 & \text{if } n \equiv 5 \pmod 8 \\ 1 & \text{if } n \equiv 6 \pmod 8 \\ 3 & \text{if } n \equiv 7 \pmod 8 \end{cases} \tag{3.2}$$

Similarly, $\overline{O}(n)$ is true if and only if $\dim H^0(Y_n, \mathcal{I}_{L_1 \cup Z}(1,2)) = 0$.

Lemma 3.11. *$\underline{O}(n)$ and $\overline{O}(n)$ are true if $n \geq 8$.*

Proof. We first show that $\underline{O}(n)$ is true for every $n \geq 8$. The proof is by eight-step induction. By Remark 3.10, it suffices to prove that $H^0(Y_n, \mathcal{I}_{L_1 \cup Z}(1,2))$ has the expected dimension.

Specialize $p_1, \ldots, p_{t(n)}$ to L_2 and $q_1 \ldots, q_{\underline{s}(n-16)}$ to $L_1 \cap L_2$. Then we obtain the following short exact sequence:

$$0 \to \mathcal{I}_{L_1 \cup L_2 \cup Z}(1,2) \to \mathcal{I}_{L_1 \cup Z}(1,2) \to \mathcal{I}_{(L_1 \cup Z) \cap L_2, L_2}(1,2) \to 0.$$

By Proposition 3.7, $\dim H^0(Y_n, \mathcal{I}_{L_1 \cup L_2 \cup Z}(1,2)) = 0$. We can check that $\underline{O}(n)$ is true for each $i \in \{8, \ldots, 15\}$ in the same way as described in Remark 3.9. Thus it follows from the induction hypothesis that $\underline{O}(n)$ is true for every $n \geq 8$. We can prove that $\overline{O}(n)$ is true for every $n \geq 8$ by replacing $\underline{s}(n)$ by $\overline{s}(n)$. □

Theorem 3.12. *$T(n)$ is true for every $n \geq 0$.*

Proof. Clearly the statement $T(0)$ holds. It is also known that $T(n)$ is true for $n = 1$ (see [11] for the proof). Thus we may assume that $n \geq 2$. To prove Theorem 3.12, we only need to show that the sub-statements $T(n; \underline{s}(n))$ and $T(n; \overline{s}(n))$ are both true (see Remark 2.4). Here we only prove that $T(n; \underline{s}(n))$ is true, because the remaining case follows the same path. The proof is by eight-step induction. One can directly prove that $T(n; \underline{s}(n))$ is true for every $i \in \{2, \ldots, 7\}$ in the same way as described in Remark 2.2. Suppose now that $T(n-8; \underline{s}(n-8))$ is true for some $n \geq 8$. Let $p_1, \ldots, p_{\underline{s}(n)} \in Y_n$ and let $Z = \{p_1^2, \ldots, p_{\underline{s}(n)}^2\}$. Suppose that only $p_1, \ldots, p_{\underline{s}(n-8)}$ lie in L_1. Then we have the following short exact sequence:

$$0 \to \mathcal{I}_{L_1 \cup Z}(1,2) \to \mathcal{I}_Z(1,2) \to \mathcal{I}_{Z \cap L_1, L_1}(1,2) \to 0.$$

By Lemma 3.11, we have

$$\dim H^0(Y_n, \mathcal{I}_{L_1 \cup Z}(1,2)) = N(n) - N(n-8) - (2n+1) \cdot t(n+8) - 16 \cdot \underline{s}(n-8).$$

It follows from the induction hypothesis that

$$\dim H^0(L_1, \mathcal{I}_{Z \cap L_1}(1,2)) = N(n-8) - \underline{s}(n-8)[2(n-8)+1].$$

This implies that

$$\begin{aligned} \dim H^0(Y_n, \mathcal{I}_Z(1,2)) &\leq \dim H^0(Y_n, \mathcal{I}_{L_1 \cup Z}(1,2)) + \dim H^0(L_1, \mathcal{I}_{Z \cap L_1}(1,2)) \\ &= N(n) - \underline{s}(n)(2n+1). \end{aligned}$$

On the other hand, since $h_{Y_n}(Z,(1,2)) \leq \underline{s}(n)(2n+1)$, we have

$$\dim H^0(Y_n, \mathcal{I}_Z(1,2)) = N(n) - h_{Y_n}(Z,(1,2)) \geq N(n) - \underline{s}(n)(2n+1),$$

which implies that $\dim H^0(Y_n, \mathcal{I}_Z(1,2)) = N(n) - \underline{s}(n)(2n+1)$. Thus $T(n; \underline{s}(n))$ is true, which completes the proof. □

4. Segre varieties $\mathbb{P}^n \times \mathbb{P}^{n-1}$ embedded by $\mathcal{O}(1,2)$

Let V be an $(n+1)$-dimensional vector space over $\mathbb{C}$ with basis $\{e_0, \ldots, e_n\}$ and let W be an n-dimensional vector space over $\mathbb{C}$ with basis $\{f_0, \ldots, f_{n-1}\}$. We write $Y_{n,n-1}$ for $\mathbb{P}^n \times \mathbb{P}^{n-1} = \mathbb{P}(V) \times \mathbb{P}(W)$ and indicate by $X_{n,n-1}$ for the Segre-Veronese variety $Y_{n,n-1}$ embedded by the morphism given by $\mathcal{O}(1,2)$. The goal of this section is to prove that $X_{n,n-1}$ is non-defective unless $n = 4$. We can prove it almost in the same way as in Section 3. However, slight modifications will be need.

For each $i \in \{1,2,3\}$, let $S_i = \{e_{n-2i+1}, e_{n-2(i-1)}\}$, let $T_i = \{f_{n-2i}, f_{n-2i+1}\}$, let $V_i = \langle S \setminus S_i \rangle$ and let $W_i = \langle T \setminus T_i \rangle$. If $n \geq 7$, then the subspace $\sum_{i=1}^3 V_i \otimes S_2(W_i)$ of $V \otimes S_2(W)$ has dimension

$$3N(n-2,n-3) - 3N(n-4,n-5) + N(n-6,n-7) = N(n,n-1) - 24.$$

We denote by L_i the subvariety $\mathbb{P}(V_i) \times \mathbb{P}(W_i)$ of $Y_{n,n-1}$. For each $i \in \{1,2,3\}$, the affine cone $T_p(X_{n,n-1})$ over the tangent space to $X_{n,n-1}$ at $p = [v \otimes w^2] \in X_{n,n-1}$ is congruent to $\langle S_i \rangle \otimes w^2 + v \otimes w \langle T_i \rangle$ modulo $V_i \otimes S_2(W_i)$, and so $T_p(X_{n,n-1}) + V_i \otimes S_2(W_i)$ is expected to have dimension $N(n-2,n-3)+4$.

Definition 4.1. Let $n \geq 7$. For each $i \in \{1,2,3\}$, let $p_{i,1}$ and $p_{i,2}$ be general points of L_i. Let $A_{n,n-1}$ be the following subspace of $V \otimes S_2(W)$:

$$\sum_{i=1}^3 \left[L_i + T_{p_{i,1}}(X_{n,n-1}) + T_{p_{i,2}}(X_{n,n-1}) \right].$$

Then we have the following inequality:

$$\begin{aligned} \dim A_{n,n-1} &\leq 3N(n-2,n-3) - 3N(n-4,n-5) + N(n-6,n-7) + 3(4+4) \\ &= N(n,n-1) - 24 + 24 \\ &= N(n,n-1). \end{aligned}$$

We say that *the statement* $R(n,n-1)$ *is true* if equality holds. Note that $R(n,n-1)$ is true if and only if

$$\dim H^0(Y_{n,n-1}, \mathcal{I}_{L_1 \cup L_2 \cup L_3 \cup Z}(1,2)) = 0,$$

where $Z = \{p_{i,1}^2, p_{i,2}^2\}_{1 \leq i \leq 3}$.

Proposition 4.2. *$R(n,n-1)$ is true for every $n \geq 7$.*

Proof. The proof is very similar to that of Proposition 3.4. For each $i \in \{1,2,3\}$, let $p_{i,1}, p_{i,2}$ be general points of L_i and let $Z = \{p_{i,1}^2, p_{i,2}^2\}_{1 \leq i \leq 3}$. For simplicity, we denote $L_1 \cup L_2 \cup L_3$ by L. It suffices to show $H^0(Y_n, \mathcal{I}_{L \cup Z}(1,2)) = 0$.

Let V_K be the subspace of V spanned by S_1, S_2 and S_3, let W_K be the subpsace of W spanned by T_1, T_2 and T_3 and let $K = \mathbb{P}(V_K) \times \mathbb{P}(W_K)$. It is straightforward to show that $V \otimes S_2(W) = V_K \otimes S_2(W_K) + \sum_{i=1}^3 V_i \otimes S_2(W_i)$. In another word, we have $H^0(Y_n, \mathcal{I}_{L \cup K}(1,2)) = 0$. Let V_H be a general subspace of V of dimension greater than or equal to 8 that contains V_K, let W_H be a general subspace of W of

dimension greater than or equal to 7 that contains W_K and let $H = \mathbb{P}(V_H)\otimes\mathbb{P}(W_H)$. Then it follows from the above equality that $H^0(Y_{n,n-1}, \mathcal{I}_{L\cup H}(1,2)) = 0$.

Specializing $p_{i,1}$ and $p_{i,2}$ to H, we get the following exact sequence:

$$0 \to \mathcal{I}_{L\cup Z\cup H}(1,2) \to \mathcal{I}_{L\cup Z}(1,2) \to \mathcal{I}_{(L\cup Z)\cap H,H}(1,2) \to 0.$$

Since $\dim H^0(Y_{n,n-1}, \mathcal{I}_{L\cup H}(1,2)) \geq \dim H^0(Y_{n,n-1}, \mathcal{I}_{L\cup Z\cup H}(1,2))$, we have $H^0(Y_{n,n-1}, \mathcal{I}_{L\cup Z\cup H}(1,2)) = 0$. It can be directly check that $R(7)$ is true. Thus we completed the proof. □

Definition 4.3. Let $n \geq 5$. Note that $\sum_{i=1}^{2} V_i \otimes S_2(W_i)$ has dimension

$$2N(n-2,n-3) - N(n-4,n-5) = N(n,n-1) - (12n-16).$$

Let $\underline{s}(n,n-1) = \lfloor N(n,n-1)/2n \rfloor$ and let $\overline{s}(n,n-1) = \lceil N(n,n-1)/2n \rceil$. Then it is not hard to show that

$$\begin{aligned} t(n,n-1) &= \underline{s}(n-2,n-3) - \underline{s}(n-4,n-5) \\ &= \overline{s}(n-2,n-3) - \overline{s}(n-4,n-5) \\ &= n-2. \end{aligned}$$

Let p_1 and p_2 be general points of $Y_{n,n-1}$ and let $p_{i,1}, \ldots, p_{i,t(n,n-1)}$ be general points of L_i for each $i \in \{1,2\}$. Denote by $B_{n,n-1}$ the following subspace of $V \otimes S_2(W)$:

$$\sum_{i=1}^{2} \left[L_i + T_{p_i}(X_{n,n-1}) + \sum_{j=1}^{t(n,n-1)} T_{p_{i,j}}(X_{n,n-1}) \right].$$

Then we have the following inequality:

$$\begin{aligned} \dim B_{n,n-1} &\leq 2N(n-2,n-3) - N(n-4,n-5) + 2\cdot 2n + 2\cdot 4\cdot (n-2) \\ &= N(n,n-1) - (12n-16) + 12n - 16 \\ &= N(n,n-1). \end{aligned}$$

We say that *the statement* $Q(n,n-1)$ *is true* if equality holds. Note that if

$$Z = \{p_{1,1}^2, \ldots, p_{1,t(n,n-1)}^2, p_{2,1}^2, \ldots, p_{2,t(n,n-1)}^2, p_1^2, p_2^2\},$$

then $Q(n,n-1)$ is true if and only if $\dim H^0(Y_{n,n-1}, \mathcal{I}_{L_1\cup L_2\cup Z}(1,2)) = 0$.

Proposition 4.4. *$Q(n,n-1)$ is true for every $n \geq 5$.*

Proof. The proof is by two-step induction on n. The initial cases can be directly shown. Suppose now that $Q(n-2,n-3)$ is true for some n. Let L_3 be the subvariety of $Y_{n,n-1}$ as defined above. Then $L_i \cap L_3 = Y_{n-4,n-5}$ for each $i \in \{1,2\}$. Specialize $p_{i,1}, \ldots, p_{i,t(n,n-1)-2}$ to $L_i \cap L_3$ and p_1, p_2 to L_3. Consider the short exact sequence

$$0 \to \mathcal{I}_{L_1\cup L_2\cup L_3\cup Z}(1,2) \to \mathcal{I}_{L_1\cup L_2\cup Z}(1,2) \to \mathcal{I}_{(L_1\cup L_2\cup Z)\cap L_3,L_3}(1,2) \to 0.$$

Proposition 4.2 implies that $H^0(Y_{n,n-1}, \mathcal{I}_{L_1\cup L_2\cup L_3\cup Z}(1,2)) = 0$. Therefore it follows from the induction hypothesis that $H^0(Y_{n,n-1}, \mathcal{I}_{L_1\cup L_2\cup Z}(1,2)) = 0$, which completes the proof of Proposition 4.4. □

Definition 4.5. Let $n \geq 3$. Keeping in mind the notation of Definition 4.3, let $p_1, \ldots, p_{t(n+2,n+1)}$ be general points of $Y_{n,n-1}$ and let $q_1, \ldots, q_{\underline{s}(n-2,n-3)}$ be general points of L_1. We write $\underline{C}_{n,n-1}$ for the following subspace of $V \otimes S_2(W)$:

$$L_1 + \sum_{i=1}^{t(n+2,n+1)} T_{p_i}(X_{n,n-1}) + \sum_{i=1}^{\underline{s}(n-2,n-3)} T_{q_i}(X_{n,n-1}). \tag{4.1}$$

Then we have the following inequality:

$$\begin{aligned} \dim \underline{C}_{n,n-1} &\leq N(n-2,n-3) + 2n \cdot t(n+2,n+1) + 4 \cdot \underline{s}(n-2,n-3) \\ &= N(n,n-1) - (3n^2 - 2n + 1) + 2n^2 + 4 \cdot \underline{s}(n-2,n-3) \\ &= N(n,n-1) - (n^2 - 2n + 1) + 4 \cdot \underline{s}(n-2,n-3). \end{aligned}$$

Note that

$$4 \cdot \underline{s}(n-2,n-3) = \begin{cases} n^2 - 2n & \text{if } n \text{ is even} \\ (n-1)^2 & \text{if } n \text{ is odd.} \end{cases}$$

Thus

$$\dim \underline{C}_{n,n-1} \leq \begin{cases} N(n,n-1) - 1 & \text{if } n \text{ is even} \\ N(n,n-1) & \text{if } n \text{ is odd.} \end{cases}$$

We say that *the statement* $\underline{O}(n,n-1)$ *is true* if equality holds. To prove that $\underline{O}(n,n-1)$ is true, it is enough to show that

$$\dim H^0(Y_{n,n-1}, \mathcal{I}_{L_1 \cup Z}(1,2)) = \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd,} \end{cases}$$

where $Z = \{p_1^2, \ldots, p_{t(n+2,n+1)}^2, p_{1,1}^2, \ldots, p_{1,s(n-2,n-3)}^2\}$.

Let $\overline{C}_{n,n-1}$ be the subspace obtained from (4.1) by replacing $\underline{s}(n-2,n-3)$ by $\overline{s}(n-2,n-3)$. We say that $\overline{O}(n,n-1)$ is true if $\overline{C}_n$ has the expected dimension, or if $\dim H^0(Y_{n,n-1}, \mathcal{I}_{L_1 \cup Z}(1,2)) = 0$.

Lemma 4.6. *Let* $n \geq 3$.

(i) $\underline{O}(n,n-1)$ *is true unless* $n = 4$.
(ii) $\overline{O}(n,n-1)$ *is true.*

Proof. Here we only show (i), because (ii) follows the same path. Let k be the integer such that

$$n = \begin{cases} 2k & n \text{ is even} \\ 2k+1 & n \text{ is odd.} \end{cases}$$

The proof is by induction on k. One can directly check that $\underline{O}(n,n-1)$ is true if $n = 3$ and 6. Suppose that $\underline{O}(n-2,n-3)$ is true for some n. Let $p_1, \ldots, p_{t(n+2,n+1)}$ be general points of $Y_{n,n-1}$ and let $q_1, \ldots, q_{\underline{s}(n-2,n-3)}$ be general points of L_1. Suppose that $p_1, \ldots, p_{\underline{t}(n,n-1)} \in L_2 \backslash (L_1 \cap L_2)$ and that $q_1, \ldots, q_{\underline{s}(n-4,n-3)} \in L_1 \cap L_2$. Consider the short exact sequence

$$0 \to \mathcal{I}_{L_1 \cup L_2 \cup Z}(1,2) \to \mathcal{I}_{L_1 \cup Z}(1,2) \to \mathcal{I}_{(L_1 \cup Z) \cap L_2, L_2}(1,2) \to 0. \tag{4.2}$$

By Proposition 4.4, we have $H^0(Y_{n,n-1}, \mathcal{I}_{L_1 \cup L_2 \cup Z}(1,2)) = 0$. Thus it follows from the induction hypothesis that

$$H^0(Y_{n,n-1}, \mathcal{I}_{L_1 \cup Z}(1,2)) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even,} \end{cases}$$

from which the lemma follows. □

Theorem 4.7. *$T(n,n-1;s)$ is true unless $(n,s)=(4,6)$. In particular, $T(n,n-1)$ is true except for $n=4$.*

Proof. It is enough to prove that both $T(n,n-1;\underline{s}(n,n-1))$ and $T(n,n-1;\overline{s}(n,n-1))$ are true. Suppose first that n is odd, i.e., $n=2k+1$ for some k. Then $\underline{s}(n,n-1)=\overline{s}(n,n-1)$. Thus it suffices to show that $T(n,n-1;\underline{s}(n,n-1))$ is true. The proof is by induction on k. Obviously, $T(1,0;1)$ is true. Assume that $T(n-2,n-3;\underline{s}(n-2,n-3))$ is true for some odd integer n. We use this assumption to prove that $T(n,n-1;\underline{s}(n,n-1))$ is true.

Recall that $\underline{s}(n,n-1)-\underline{s}(n-2,n-3)=n$. Let $p_1,\dots,p_n$ be general points of $Y_{n,n-1}$, let $q_1,\dots,q_{\underline{s}(n-2,n-3)}$ be general points of L_1 and let $Z=\{p_1^2,\dots,p_n^2,q_1^2,\dots,q_{\underline{s}(n-2,n-3)}^2\}$. Then we have the following exact sequence:

$$0\to\mathcal{I}_{L_1\cup Z}(1,2)\to\mathcal{I}_Z(1,2)\to\mathcal{I}_{Z\cap L_1,L_1}(1,2)\to 0.$$

By Lemma 4.6, we have $H^0(Y_{n,n-1},\mathcal{I}_{L_1\cup Z}(1,2))=0$. Thus it follows from the induction hypothesis that $H^0(Y_{n,n-1},\mathcal{I}_Z(1,2))=0$. Thus $T(n,n-1;\underline{s}(n,n-1))$ is true.

Suppose next that n is even, i.e., $n=2k$ for some k. We first prove that $T(n,n-1;\underline{s}(n,n-1))$ is true by induction on k. Note that $\underline{s}(6,5)=12$. As described in Remark 2.2, one can check that $T(6,5;12)$ is true. Suppose now that $T(n-2,n-3;\underline{s}(n-2,n-3))$ is true for some $n\geq 8$. Recall that $\underline{s}(n,n-1)-\underline{s}(n-2,n-3)=n$. Let $p_1,\dots,p_n$ be general points of $Y_{n,n-1}$, let $q_1,\dots,q_{\underline{s}(n-2,n-3)}$ be general points of L_1 and let $Z=\{p_1^2,\dots,p_n^2,q_1^2,\dots,q_{\underline{s}(n-2,n-3)}^2\}$. Then we have the following exact sequence:

$$0\to\mathcal{I}_{L_1\cup Z}(1,2)\to\mathcal{I}_Z(1,2)\to\mathcal{I}_{Z\cap L_1,L_1}(1,2)\to 0.$$

Note that

$$\dim H^0(Y_{n,n-1},\mathcal{I}_Z(1,2))\geq N(n,n-1)-2n\cdot\underline{s}(n,n-1)=n/2.$$

In order to prove that $T(n,n-1;\underline{s}(n,n-1))$ is true, it is sufficient to show that $\dim H^0(Y_{n,n-1},\mathcal{I}_Z(1,2))\leq n/2$. Since $T(n-2,n-3;\underline{s}(n-2,n-3))$ is true by induction hypothesis, we have $\dim H^0(\mathcal{I}_{Z\cap L_1,L_1}(1,2))=(n-2)/2$. By Lemma 4.6, we get $\dim H^0(Y_{n,n-1},\mathcal{I}_{L_1\cup Z}(1,2))=1$, from which it follows that

$$\begin{aligned}\dim H^0(Y_{n,n-1},\mathcal{I}_Z(1,2)) &\leq \dim H^0(\mathcal{I}_{Z\cap L_1,L_1}(1,2))+\dim H^0(Y_{n,n-1},\mathcal{I}_{L_1\cup Z}(1,2))\\ &= (n-2)/2+1\\ &= n/2.\end{aligned}$$

It remains only to prove that $T(2,1;2)$ is true and that $T(4,3;5)$ is true. This can be done in the same way as described in Remark 2.2.

One can prove that $T(n,n-1;\overline{s}(n,n-1))$ is true for every even number $n\geq 2$ in a similar way, and so we omit it. □

5. Double Induction

Let V be an $(m+1)$-dimensional vector space over $\mathbb{C}$ with basis $\{e_0,\dots,e_m\}$ and let W be an $(n+1)$-dimensional vector space over $\mathbb{C}$ with basis $\{f_0,\dots,f_n\}$. As in Section 2, we write $Y_{m,n}$ for $\mathbb{P}^m\times\mathbb{P}^n=\mathbb{P}(V)\times\mathbb{P}(W)$ and $X_{m,n}$ for the Segre-Veronese variety obtained from $Y_{m,n}$ by embedding in $\mathbb{P}^{N(m,n)-1}$ by the morphism given by $\mathcal{O}(1,2)$, where $N(m,n)=(m+1)\binom{n+2}{2}$. Suppose that $n\geq m-1$. The goal

of this section is to provide two functions $s_1(m,n) \leq s_2(m,n)$ such that $T(m,n;s)$ is true either if $s \leq s_1(m,n)$ or if $s \geq s_2(m,n)$.

For each $i \in \{1,2\}$, let $S_i = \{f_{2(i-1)}, \ldots, f_{n+2(i-2)}\}$, let $W_i = \langle S_i \rangle$ and let $L_i = \mathbb{P}(V) \times \mathbb{P}(W_i)$.

Definition 5.1. Let s be a positive integer with $s \geq m+1$, let $p_1, \ldots, p_{s-(m+1)}$ be general points of L_1 and let $q_1, \ldots, q_{m+1}$ be general points of $Y_{m,n}$. We denote by $V_{m,n,s}$ the following subspace of $V \otimes S_2(W)$:

$$V \otimes S_2(W_1) + \sum_{i=1}^{s-(m+1)} T_{p_i}(X_{m,n}) + \sum_{i=1}^{m+1} T_{q_i}(X_{m,n}).$$

Note that the dimension of $V_{m,n,s}$ is expected to be

$$\min\left\{N(m,n-2) + 2[s-(m+1)] + (m+1)(m+n+1), N(m,n)\right\}. \tag{5.1}$$

We say that *the statement* $R(m,n;s)$ *is true* if $V_{m,n,s}$ has the expected dimension.

Remark 5.2. Let $k = \lfloor n/2 \rfloor$ and let

$$\mathfrak{s}(m,n) = \begin{cases} (m+1)k - (m-2)(m+1)/2 & \text{if } n \text{ is even;} \\ (m+1)k - (m-3)(m+1)/2 & \text{if } m \text{ and } n \text{ are odd;} \\ (m+1)k - [(m-3)(m+1)+1]/2 & \text{if } m \text{ is even and if } n \text{ is odd.} \end{cases}$$

It was shown in [1] that $R(m,n;\mathfrak{s}(m,n))$ is true, i.e.,

$$\dim V_{m,n,\mathfrak{s}(m,n)} = \begin{cases} N(m,n) & \text{if } n \text{ is even of if } m \text{ and } n \text{ are both odd;} \\ N(m,n)-1 & \text{otherwise.} \end{cases}$$

By (5.1), this implies that if n is even or if m and n are both odd, then $R(m,n;s)$ is true for every s; while if m is even and if n is odd, then $R(m,n;s)$ is true for every $s \leq \mathfrak{s}(m,n)$.

Definition 5.3. For each $i \in \{1,2\}$, let $p_{i,1}, \ldots, p_{i,m+1}$ be general points of L_i. The subspace

$$\sum_{i=1}^{2} \left[V \otimes S_2(W_i) + \sum_{j=1}^{m+1} T_{p_{i,j}}(X_{m,n}) \right]$$

of $V \otimes S_2(W)$ will be denoted by $W_{m,n}$. It is expected that $\dim W_{m,n} = N(m,n)$. We say that *the statement* $Q(m,n)$ *is true* if $W_{m,n}$ has the expected dimension.

Remark 5.4. In [1], it has already been proved that if $n \geq 3$, then $Q(m,n)$ is true for any m (see Proposition 3.11 in [1] for more details).

Lemma 5.5. *Suppose that s is a positive integer larger than or equal to $2(m+1)$. If $R(m,n-2;s-(m+1))$ are true, then so is $R(m,n;s)$.*

Proof. Let $p_1, \ldots, p_{s-(m+1)} \in L_1$ and let $q_1, \ldots, q_{m+1} \in Y_{m,n} \setminus L_1$. Suppose that $p_1, \ldots, p_{s-2(m+1)} \in L_1 \cap L_2$, $p_{s-2m-1} \ldots, p_{s-(m+1)} \in L_1 \setminus L_1 \cap L_2$ and $q_1, \ldots, q_{m+1} \in L_2$. Then we have a short exact sequence

$$0 \to \mathcal{I}_{Z \cup L_1 \cup L_2}(1,2) \to \mathcal{I}_{Z \cup L_1}(1,2) \to \mathcal{I}_{(Z \cup L_1) \cap L_2, L_2}(1,2) \to 0.$$

By Remark 5.4, we have $\dim H^0(Y_{m,n}, \mathcal{I}_{Z \cup L_1 \cup L_2}(1,2)) = 0$. Thus the following inequality holds:

$$\dim H^0(Y_{m,n}, \mathcal{I}_{Z \cup L_1}(1,2)) \leq \dim H^0(L_2, \mathcal{I}_{(Z \cup L_1) \cap L_2, L_2}(1,2)).$$

Hence if $R(m,n-2;s-(m+1))$ is true, then so is $R(m,n;s)$. □

Lemma 5.6. *Let m be a positive integer. Then $R(m,m;s)$ is true for every positive integer $s \geq m+1$.*

Proof. See Proposition 3.4 in [1] for the proof. □

Lemma 5.7. *Let m be a positive integer. Then $R(m,m+1;s)$ is true for every positive integer $s \geq m+1$.*

Proof. It has already been shown that $R(m,n;s)$ is true if $s \leq \mathfrak{s}(m,m+1)$ (see Proposition 3.5 in [1]). Thus it remains to show that the statement holds for every $s > \mathfrak{s}(m,m+1)$. To do so, it is enough to prove that $R(m,m+1;\mathfrak{s}(m,m+1)+1)$ is true.

We only prove Proposition 5.7 for m even, because the remaining case can be proved in the same way. Note that if m is even, then $\mathfrak{s}(m,m+1)+1 = 3m/2+2$. Let $p_0,\ldots,p_{m/2} \in L_1$ and let $q_1,\ldots,q_{m+1} \in Y_{m,m+1} \setminus L_1$, let $W' = \langle f_1,\ldots,f_{m+1}\rangle$ and let $Y_{m,m}$ be the subvariety $\mathbb{P}(V)\times\mathbb{P}(W')$ of $Y_{m,m+1}$. Note that $L_1 \cap Y_{m,m} = Y_{m,m-2}$. Specialize $q_1,\ldots,q_{m+1}$ on $Y_{m,m} \setminus L_1$. Suppose that $p_0,\ldots,p_{m/2} \notin Y_{m,m}$. Let $Z = \{p_0^2,\ldots,p_{m/2}^2,q_1^2,\ldots,q_{m+1}^2\}$. Then we have the following short exact sequence:

$$0 \to \mathcal{I}_{Z\cup L_1\cup Y_{m,m}}(1,2) \to \mathcal{I}_{Z\cup L_1}(1,2) \to \mathcal{I}_{(Z\cup L_1)\cap Y_{m,m},Y_{m,m}}(1,2) \to 0.$$

By Proposition 5.6, $R(m,m;m+1)$ is true. Thus

$$\dim H^0(\mathcal{I}_{(Z\cup L_1)\cap Y_{m,m},Y_{m,m}}(1,2)) = 0.$$

So we have

$$\dim H^0(Y_{m,m+1},\mathcal{I}_{Z\cup L_1\cup Y_{m,m}}(1,2)) = \dim H^0(Y_{m,m+1},\mathcal{I}_{Z\cup L_1}(1,2)).$$

Hence we only need to prove that $\dim H^0(Y_{m,m+1},\mathcal{I}_{Z\cup L_1\cup Y_{m,m}}(1,2)) = 0$.

Let $X'_{m,m+1}$ be the Segre-Veronese variety obtained by embedding $Y_{m,m+1}$ into $\mathbb{P}^{(m+1)(m+2)-1}$ by the morphism given by $\mathcal{O}(1,1)$ and let

$$\widetilde{Z} = \{p_0^2,\ldots,p_{m/2}^2,q_1,\ldots,q_{m+1}\}.$$

For each $i \in \{1,\ldots,m+1\}$, let $q_i = [v'_i \otimes w'_i] \in X'_{m,n}$. Since

$$H^0(Y_{m,m+1},\mathcal{I}_{Z\cup L_1\cup Y_{m,m}}(1,2)) \simeq H^0(Y_{m,m+1},\mathcal{I}_{\widetilde{Z}\cup L_1}(1,1)),$$

the condition that $\dim H^0(Y_{m,m+1},\mathcal{I}_{Z\cup L_1\cup Y_{m,m}}(1,2)) = 0$ is equivalent to the condition that the following subspace of $V\otimes W$ has dimension $(m+1)(m+2)$:

$$V\otimes W_1 + \sum_{i=0}^{m/2} T_{p_i}(X'_{m,m+1}) + \sum_{i=1}^{m+1}\langle v'_i\otimes w'_i\rangle.$$

Since $p_i \in L_1$ for each $i \in \{0,\ldots,m/2\}$, there are $v_i \in V$ and $w_i \in W_1$ such that $p = [v_i \otimes w_i]$. Notice that $T_{p_i}(X'_{m,m+1}) = V\otimes w_i + v_i\otimes W$. Thus $T_{p_i}(X'_{m,m+1})$ is congruent to $v_i\otimes\langle f_m,f_{m+1}\rangle$ modulo $V\otimes W_1$, which implies that

$$V\otimes W_1 + T_{p_i}(X'_{m,m+1}) = (V\otimes f_0)\oplus\left[V\otimes(W_1\cap W') + \sum_{i=0}^{m/2} v_i\otimes\langle f_m,f_{m+1}\rangle\right].$$

Thus we obtain the following equality:

$$V \otimes W_1 + \sum_{i=0}^{m/2} T_{p_i}(X'_{m,m+1}) + \sum_{i=1}^{m+1} \langle v'_i \otimes w'_i \rangle$$
$$= (V \otimes f_0) \oplus \left[V \otimes (W_1 \cap W') + \sum_{i=0}^{m/2} v_i \otimes \langle f_m, f_{m+1} \rangle + \sum_{i=1}^{m+1} \langle v'_i \otimes w'_i \rangle \right].$$

Note that $T_1 = \{ e_i \otimes f_0 \mid 0 \leq i \leq m \}$ and $T_2 = \{ e_i \otimes f_j \mid 0 \leq i \leq m, 1 \leq j \leq m-1 \}$ are bases for $V \otimes f_0$ and $V \otimes (W_1 \cap W')$ respectively. Let $v_i = e_i$ for every $i \in \{0, \ldots, m/2\}$. Then $T_3 = \{ e_i \otimes f_j \mid 0 \leq i \leq m/2, m \leq j \leq m+1\}$ is a spanning set for $\sum_{i=0}^{m/2} v_i \otimes \langle f_m, f_{m+1} \rangle$. Let T_4 be the set of vectors of the standard basis for $V \otimes W$ which are not included in the set $T_1 \cup T_2 \cup T_3$. Then T_4 consists of m distinct non-zero vectors. Choose these m elements of T_4 as $v'_i \otimes w'_i$'s (we can choose the remaining one arbitrarily). Then $\bigcup_{i=1}^{4} T_i$ spans a vector space of dimension $(m+1)(m+2)$. Thus we completed the proof. □

Theorem 5.8. *Suppose that $n \geq m$. Then $R(m, n; s)$ is true for every positive integer $s \geq m+1$.*

Proof. In [1], it has already been proved that $R(m, n; s)$ is true if $s \leq \mathfrak{s}(m, n)$. Thus we only need to prove that $R(m, n; s)$ is true for any $s > \mathfrak{s}(m, n)$. To do so, it is sufficient to show that $R(m, n; \mathfrak{s}(m, n) + 1)$ is true.

The proof is by induction on n. By Lemmas 5.6 and 5.7, $R(m, m; \mathfrak{s}(m, m) + 1)$ and $R(m, m+1; \mathfrak{s}(m, m+1)+1)$ are true. Assume that $R(m, n-2; \mathfrak{s}(m, n-2)+1)$ is true for some $n \geq m+2$. Additionally, we may assume that $n \geq 3$. Since $\mathfrak{s}(m, n) - \mathfrak{s}(m, n-2) = m+1$, the theorem immediately follows from Lemma 5.5 and Remark 5.4. □

Lemma 5.9. *Suppose that $n \geq m-1$. Let s be a positive integer with $s \geq m+1$.*
(i) *Suppose that the following inequalities hold:*

$$\begin{aligned} N(m, n-2) + 2[s-(m+1)] + (m+1)(m+n+1) &\leq N(m, n); \\ [s-(m+1)]\,(m+n-1) &\leq N(m, n-2); \\ s(m+n+1) &\leq N(m, n). \end{aligned}$$

If $T(m, n-2; s-(m+1))$ is true, then $T(m, n; s)$ is also true.
(ii) *Suppose that the following inequalities hold:*

$$\begin{aligned} N(m, n-2) + 2[s-(m+1)] + (m+1)(m+n+1) &\geq N(m, n); \\ [s-(m+1)]\,(m+n-1) &\geq N(m, n-2); \\ s(m+n+1) &\geq N(m, n). \end{aligned}$$

If $T(m, n-2; s-(m+1))$ is true, then $T(m, n; s)$ is also true.

Proof. Here we only show (i), because (ii) can be proved in the same way. Let s be a positive integer with $s \geq m+1$, let $p_1, \ldots, p_s$ be general points of $Y_{m,n}$ and let $Z = \{p_1^2, \ldots, p_s^2\}$. To prove (i), it suffices to show that $\dim H^0(Y_{m,n}, \mathcal{I}_Z(1,2)) = N(m, n) - s(m+n+1)$. Since

$$\begin{aligned} \dim H^0(Y_{m,n}, \mathcal{I}_Z(1,2)) &= \dim H^0(Y_{m,n}, \mathcal{O}(1,2)) - h_{Y_{m,n}}(Z, (1,2)) \\ &\geq N(m, n) - s(m+n+1), \end{aligned}$$

it is enough to prove that $\dim H^0(Y_{m,n}, \mathcal{I}_Z(1,2)) \leq N(m,n) - s(m+n+1)$. Suppose now that $p_1, \ldots, p_{s-(m+1)}$ lie on L_1. Then we have the following short exact sequence:

$$0 \to \mathcal{I}_{L_1 \cup Z}(1,2) \to \mathcal{I}_Z(1,2) \to \mathcal{I}_{Z \cap L_1, L_1}(1,2) \to 0.$$

Taking the cohomology, we obtain

$$\dim H^0(Y_{m,n}, \mathcal{I}_Z(1,2)) \leq \dim H^0(Y_{m,n}, \mathcal{I}_{L_1 \cup Z}(1,2)) + \dim H^0(\mathcal{I}_{Z \cap L_1, L_1}(1,2)).$$

In Theorem 5.8, it was shown that $R(m,n;s)$ is true. Thus we get

$$\begin{aligned} & \dim H^0(Y_{m,n}, \mathcal{I}_{L_1 \cup Z}(1,2)) \\ = \;& N(m,n) - [N(m,n-1) + 2\{s-(m+1)\} + (m+1)(m+n+1)]. \end{aligned}$$

By the assumption that $T(m,n-1;s-(m+1))$ is true, the following equality holds:

$$\dim H^0(L_1, \mathcal{I}_{Z \cap L_1}(1,2)) = N(m,n) - [s-(m+1)](m+n-1).$$

We therefore have the inequality

$$\begin{aligned} \dim H^0(Y_{m,n}, \mathcal{I}_Z(1,2)) &\leq \dim H^0(Y_{m,n}, \mathcal{I}_{L_1 \cup Z}(1,2)) + \dim H^0(\mathcal{I}_{Z \cap L_1, L_1}(1,2)) \\ &= N(m,n) - s(m+n+1), \end{aligned}$$

which completes the proof. □

Recall that $\overline{s}(m,m) = \lceil N(m,m)/(2m+1) \rceil$ and $\overline{s}(m,m-1) = \lceil N(m,m-1)/2m \rceil$.

Theorem 5.10. *Suppose that $n \geq m-1$.*
(i) *Let $k = \lfloor n/2 \rfloor$ and let $s_1(m,n)$ be the function defined by*

$$s_1(m,n) = \begin{cases} (m+1)k - (m-2)(m+1)/2 & \text{if } n \text{ is even;} \\ (m+1)k - (m-3)(m+1)/2 & \text{if } m \text{ and } n \text{ are odd;} \\ (m+1)k - [(m-3)(m+1)+1]/2 & \text{otherwise.} \end{cases}$$

Then $T(m,n;s)$ is true for any $s \leq s_1(m,n)$.
(ii) *Let $s_2(m,n)$ be the function defined by*

$$s_2(m,n) = \begin{cases} \overline{s}(m,m) + (m+1)(n-m)/2 & \text{if } 2|(n-m) \\ \overline{s}(m,m-1) + (m+1)(n-m+1)/2 & \text{if } 2 \not| (n-m), \end{cases}$$

Then $T(m,n;s)$ is true for any $s \geq s_2(m,n)$.

Proof. Here we only prove (ii), because the remaining case follows exactly the same path (see also Theorem 3.13 in [1]). To show (ii), it is sufficient to prove that $T(m,n;s_2(m,n))$ is true. The proof is by two-step induction on n. The following inequalities immediately follows from the definition of $s_2(m,n)$:

$$\begin{aligned} N(m,n-2) + 2[s_2(m,n) - (m+1)] + (m+1)(m+n+1) &\geq N(m,n); \\ [s_2(m,n) - (m+1)](m+n-1) &\geq N(m,n-2); \\ s_2(m,n)(m+n+1) &\geq N(m,n). \end{aligned}$$

Since, by definition, $s_2(m,m) = \overline{s}(m,m)$ and $s_2(m,m-1) = \overline{s}(m,m-1)$, $T(m,m;s_2(m,m))$ and $T(m,m-1;s_2(m,m-1))$ are true by Theorems 3.12 and 4.7. Assume that $T(m,n-2;s_2(m,n-2))$ is true. This implies that $T(m,n-2;s_2(m,n)-(m+1))$ is true, because $s_2(m,n)-s_2(m,n-2) = m+1$. Thus it follows from Theorem 5.8 and Proposition 5.9 that $T(m,n;s_2(m,n))$ is true. □

Remark 5.11. Let m and n be integers with $n \geq m-1$ and let $s_1(m,n)$ be the function as defined in Theorem 5.10. Let $\underline{s}(m,n) = \left\lfloor (m+n+1)^{-1}(m+1)\binom{n+2}{2} \right\rfloor$. Then

$$\underline{s}(m,n) = \begin{cases} (m+1)\lfloor n/2 \rfloor - \frac{(m-2)(m+1)}{2} + \left\lfloor \frac{m^3-m}{2(m+n+1)} \right\rfloor & \text{if } n \text{ is even;} \\ (m+1)\lfloor n/2 \rfloor - \frac{(m-3)(m+1)}{2} + \left\lfloor \frac{m^3-m}{2(m+n+1)} \right\rfloor & \text{if } m \text{ and } n \text{ are odd;} \\ (m+1)\lfloor n/2 \rfloor - \frac{(m-3)(m+1)+1}{2} + \left\lfloor \frac{n+m^3+2}{2(m+n+1)} \right\rfloor & \text{otherwise.} \end{cases}$$

Define a function $r(m,n)$ as follows:

$$r(m,n) = \begin{cases} m^3 - 2m & \text{if } m \text{ is even and if } n \text{ is odd;} \\ \frac{(m-2)(m+1)^2}{2} & \text{otherwise.} \end{cases}$$

Then $s_1(m,n) = \underline{s}(m,n)$ if $n \geq r(m,n)$.

Remark 5.12. Let m and n be integers with $n \geq m-1$ and let $s(m,n)$ be the function defined by

$$s(m,n) = \begin{cases} (m+1)\lfloor n/2 \rfloor + 1 & \text{if } n \text{ is even;} \\ (m+1)\lfloor n/2 \rfloor + 3 & \text{otherwise.} \end{cases}$$

In [1], it was shown that if $s \geq s(m,n)$, then $T(m,n;s)$ is true. It is also straightforward to show the following inequalities:

$$(m+1)\lfloor n/2 \rfloor - \overline{s}(m,n) \geq \begin{cases} \frac{(m+1)(m^2-6m+4)}{4m+2} - 1 & \text{if 2 divides } n-m; \\ \frac{(m+1)(m-7)}{4} - 1 & \text{otherwise.} \end{cases}$$

This implies that $s(m,n) - s_2(m,n) = O(m^2)$. Thus Theorem 5.10 (ii) improves the bound presented in [1].

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