Sample solutions of selected problems with Macaulay2 (Sets 2, 3)

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Problem 4 (Set 2). Let $R$ be the polynomial ring with variables $x$ and $y$ over rational numbers $\mathbb{Q}$. 

```plaintext
i1 : R=QQ[x,y]
o1 = R
o1 : PolynomialRing
```

Define the ideal generated by $x^2$ and $xy$:

```plaintext
i2 : I=ideal(x^2,x*y)
o2 = ideal (x , x*y)
o2 : Ideal of R
```

Compute the radical of $I$:

```plaintext
i3 : radI=radical I
o3 = ideal x
o3 : Ideal of R
```

In Macaulay2, you can use the function `primaryDecomposition` to find the primary decomposition of an ideal:

```plaintext
i4 : compI=primaryDecomposition(I)
o4 = {monomialIdeal x, monomialIdeal (x , y)}
o4 : List
```

This command returns the list of primary ideals. Their intersection is equal to $I$. Let $I_1$ be the first ideal in `compI` and let $I_2$ be the second ideal in `compI`.
i5 : I1=compI#0
o5 = monomialIdeal x
o5 : MonomialIdeal of R

i6 : I2=compI#1

\(2\)
o6 = monomialIdeal (x, y)
o6 : MonomialIdeal of R

Compute the intersect, denoted by \(J\), of \(I_1\) and \(I_2\) with intersect:

i7 : J=intersect(I1,I2)

\(2\)
o7 = monomialIdeal (x, xy)
o7 : MonomialIdeal of R

This output tells you that \(I = J\).

Problem 5 (Set 2). Redefine the ideal:

i8 : I=ideal (x^2-1,y^2-4)

\(2\)
o8 = ideal (x - 1, y - 4)
o8 : Ideal of R

As in Problem 4, the primary decomposition can be computed by primaryDecomposition:

i9 : compI=primaryDecomposition(I)

-- used 0.03 seconds
-- used 0.01 seconds
-- used 0.06 seconds

o9 = {ideal (y - 2, x + 1), ideal (y - 2, x - 1), ideal (y + 2, x + 1), ideal (y + 2, x - 1)}
o9 : List
The output does not fit. So let’s count the number of components. This is given as follows:

\[ i10 : \#\text{comp}I \]

\[ o10 = 4 \]

You can see each component of \( I \) as in the previous problem. Clearly, they are maximal ideals.

**Problem 6,7 (Set 2).** Consider the following ideal:

\[ i11 : I=\text{ideal}(x^2,x*y^2,y^3) \]

\[ o11 = \text{ideal} (x^2, x*y^2, y^3) \]

\[ o11 : \text{Ideal of } R \]

Define the quotient ring \( R/I \) as follows:

\[ i12 : Q=R/I \]

\[ o12 = Q \]

\[ o12 : \text{QuotientRing} \]

Note that \( V(I) \) is zero-dimensional, because its radical is \((x, y)\). So \( Q \) is a finite-dimensional vector space. A basis for this vector space can be computed with basis:

\[ i13 : \text{basis } Q \]

\[ o13 = | 1 x y y^2 | \]

\[ o13 : \text{Matrix } Q \text{ --- } Q \]

Replacing \( I \) by \((y^2 - x^2, y^2 + x^2)\) and taking the same steps give you the answer to Problem 7.

**Problem 12 (Set 3).** The set \( V \) of the three points is defined in affine 3-space. Define the corresponding polynomial ring:
Define the corresponding ideals:

Define the corresponding ideals:

\begin{align*}
i15 : & \text{I1=ideal(x-1,y-2,z-3)} \\
o15 = & \text{ideal (x - 1, y - 2, z - 3)} \\
o15 : & \text{Ideal of S}
\end{align*}

\begin{align*}
i16 : & \text{I2=ideal(x-1,y-3,z-7)} \\
o16 = & \text{ideal (x - 1, y - 3, z - 7)} \\
o16 : & \text{Ideal of S}
\end{align*}

\begin{align*}
i17 : & \text{I3=ideal(x-2,y-3,z-5)} \\
o17 = & \text{ideal (x - 2, y - 3, z - 5)} \\
o17 : & \text{Ideal of S}
\end{align*}

Compute the intersection of these ideals:

\begin{align*}
i18 : & \text{I=intersect(I1,I2,I3)} \\
& \text{o18 = ideal \(-z - ---z + ---z - ----, - *y*z + *y*z + - z - -y - 3z + -2)}
\end{align*}

\begin{align*}
o18 = & \text{ideal (---z - ---z + ---z - ----, - *y*z + *y*z + - z - -y - 3z + -2)} \\
o18 : & \text{Ideal of S}
\end{align*}

Is this ideal radical?

\begin{align*}
i19 : & \text{I==radical I} \\
o19 = & \text{true}
\end{align*}

So I is equal to \( I(V) \).