Problem Set 14

A quick note on degree. Let $V$ be an affine variety of dimension $p$. If you intersect $V$ with a "random" hyperplane then you will get a variety of dimension $p-1$. If you intersect $V$ with $p$ random hyperplanes then you should get something which is zero dimensional. On the level of ideals, if you take the union of $I(V)$ with the ideal of $p$ random linear forms then you will get an ideal $J$. The degree of $V$ is defined to be $\dim_k(R/J)$. We can compute the dimension of $V$ by determining how many hyperplane sections are needed to reduce $V$ to a set of points.

Suppose $V$ is a smooth irreducible curve in $\mathbb{A}^3$. To each point $P \in V$ we can glue the tangent line to $V$ at $P$. In this manner, we get a surface in $\mathbb{A}^3$ called the tangent variety of $V$. Here is one strategy for computing the equation of such a surface from $I(V)$. Let $I(V) \subseteq k[x, y, z]$. Construct an ideal $J$ in $k[A, B, C, x, y, z]$ that contains all of the information of all the tangent lines at each choice of $(A, B, C) \in V$. This can be done by the following steps:

1) Construct the Jacobian matrix, $\text{Jac}(I)$, and multiply it by the transpose of the matrix $[x - A, y - B, z - C]$. Take all of the entries of the resulting product and form an ideal $L$.

2) Form the ideal $I'$ by taking the generators of $I$ and replacing $x$ with $A$, $y$ with $B$, and $z$ with $C$.

3) Now the ideal generated by $L + I'$ contains all of the information we need. Compute $G = (L + I') \cap k[x, y, z]$. $G$ is the ideal of the tangent variety.

Problem 1. Let $I = (x^2 - y, x^3 - z) \subseteq k[x, y, z]$. Then $I$ is the ideal of the twisted cubic curve in $\mathbb{A}^3$. The tangent variety of the twisted cubic is a surface and will be generated by one equation, $F$.

a) What is the degree of the twisted cubic?

b) What do you think the degree of $F$ will be?

c) Compute $F$. Were you correct?

d) Can you gain any feeling for the relationship of the degree of $F$ to the degree of $V$?

Problem 2. Repeat the previous problem for the Veronese surface in $\mathbb{A}^5$. 