Problem 1. Let $F$ and $G$ be two homogeneous polynomials in $R = k[x_0, x_1, \ldots, x_n]$. Assume that $F$ and $G$ do not share any common factors. Suppose that $A, B \in R$ and $AF + BG = 0$. Prove that there exists an $H \in R$ such that $A = -HG$ and $B = HF$. (In other words prove that every syzygy of $[F \ G]$ is of the form $H \begin{bmatrix} -G \\ F \end{bmatrix}$.)

Suppose the Hilbert Polynomial of $R/I$ is $F(X) = a_rX^r + a_{r-1}X^{r-1} + \cdots + a_1X + a_0$. Associated to $I$ is a scheme (not necessarily equidimensional). The dimension of the scheme is equal to $r$. The degree of the scheme is $a_r/r!$. If $F(X) = a_1X + a_0$ then the scheme is one dimensional and has degree $a_1$. Let $G = 1 - a_0$. If the scheme is one dimensional then $G$ is called the arithmetic genus.

Problem 2. a) Use the previous problem to determine the Hilbert Polynomial of $k[x_0, x_1, x_2, x_3]/(F, G)$ where $F$ is irreducible of degree 3 and $G$ is irreducible of degree 5.

b) When do the Hilbert Function and Hilbert Polynomial of $k[x_0, x_1, x_2, x_3]/(F, G)$ start to agree?

c) What is the arithmetic genus of the scheme defined by $F$ and $G$. (If $F$ and $G$ meet transversally at their points of intersection then we can think of the variety defined by $F$ and $G$ instead of the scheme defined by $F$ and $G$.)

Problem 3. Let $M$ be a $4 \times 4$ matrix. Let $M_{\text{top}}$ denote the top two rows of $M$. Let $M_{\text{bottom}}$ denote the bottom two rows of $M$. It was claimed in class that the determinant of $M$ could be written in terms of the $2 \times 2$ minors of $M_{\text{top}}$ and the $2 \times 2$ minors of $M_{\text{bottom}}$. Let $a_{ij}$ denote the $ij^{th}$ entry of $M$. If $M^{ij} = a_1a_{2j} - a_2a_{1j}$ and $M_{ij} = a_3a_{4j} - a_4a_{3j}$, then write the determinant of $M$ in terms of these symbols and verify that it is correct.

Problem 4. Recall that the join of two varieties $V, W \subseteq \mathbb{P}^n$ is defined to be the closure of the set of all lines joining a point of $V$ to a point of $W$. We will write this as $J(V, W)$. Let $R = k[a, b, c, d, e]$. Let $L_1$ be the line defined by the ideal $(a - b, c + d, e) \subseteq R$ and let $L_2$ be the line defined by the ideal $(a + c, b + d, a + b + e) \subseteq R$. Find generators for the ideal of $J(L_1, L_2)$.

Problem 5. Determine the equation of the secant variety to the Veronese Surface in $\mathbb{P}^5$. 