Partial differentiations.

Let $k$ be an algebraically closed field, and let $R$ denote the polynomial ring $k[x, y, z]$. Assume that $I$ is an ideal with $V(I) = \emptyset$. Then the quotient ring $Q$ of $R$ modulo $I$ is a finite-dimensional vector space over $k$. In other words, $Q$ is artinian. In this case, $Q$ has a free resolution of the following type (this is not trivial):

$$0 \rightarrow F_3 \rightarrow F_2 \rightarrow F_1 \rightarrow R \rightarrow Q \rightarrow 0,$$

where $F_i$ are free modules for each $i = 1, 2$ and 3. This ring is called \textit{arithmetically Gorenstein} if $F_3$ can be written as $R(-l)$ for some positive integer $l$ (i.e. $F_3$ has rank 1).

**Remark 1.** Let $Q$ be an artinian arithmetically Gorenstein ring over $R$. Then there is a positive integer $d$ such that $\dim_k(Q_d) = 1$ and $\dim_k(Q_i) = 0$ for all $i > d$.

Let $S$ be the polynomial ring $k[X, Y, Z]$, and let $S$ act on $R$ by partial differentiation:

$$X(x) := \partial_x(x), Y(y) := \partial_y(y) \text{ and } Z(z) := \partial_z(z).$$

Let $F$ be a single homogeneous polynomial of degree $d$ in $R$. For this $F$, denote by $I_S(F)$ the set of polynomials $G$ in $S$ satisfying $G(F) = 0$. Then $I_S(F)$ is an ideal of $S$ (why?). Consider the quotient ring $Q(F)$ of $S$ modulo $I_S(F)$. It is known that this ring is artinian and arithmetically Gorenstein.

**Remark 2.** By definition, $\dim_k(Q(F))_i = 0$ for all $i > d$. It immediately follows that $Q(F)$ is artinian. Let $G$ be a degree $d$ homogeneous polynomial in $I_S(F)$:

$$G = \sum_{i+j+k=d} a_{ijk}x^iy^jz^k.$$

Then $G(F)$ can be written as a $k$-linear combination of $a_{ijk}$’s. So the elements $G$ satisfying $G(F) = 0$ form a one-codimensional subspace in $S_d$, and hence $\dim_k(Q(F)_d) = 1$. In general, we have the following equations:

$$\dim_k(Q(F)_r) = \dim_k(Q(F)_{d-r}) \text{ for all } 0 \leq r < d/2.$$
Proposition. If \( Q \) is an artinian arithmetically Gorenstein ring of \( S \), then there is a polynomial \( F \) in \( R \) such that \( Q = S/I_S(F) \). Furthermore, such a polynomial is uniquely determined up to constants.


Let us discuss how to compute the corresponding polynomial \( F \) in \( R \) from a given artinian arithmetically Gorenstein ring \( Q \) of \( S \). From Remark 1, it follows that there is a positive integer \( d \) such that \( \dim_k(Q_d) = 1 \) and \( \dim_k(Q_i) = 0 \) for \( i > d \). Let \( I \) be the ideal in \( S \), that is obtained as the kernel of the ring homomorphism from \( S \) to \( Q \), and let \( \{f_1, \ldots, f_t\} \) be a set of generators of \( I_d \), where

\[
t = \dim_k(S_d) - \dim_k(Q_d) = \left( \frac{d + 2}{2} \right) - 1.
\]

Consider the bilinear map \( \tilde{T} \) from \( I_d \otimes_k R_d \) to \( k \) defined by \( \tilde{T}(G \otimes F) = G(F) \). Recall that this bilinear map corresponds to a linear transformation \( T \) from \( R_d \) to \((I_d)^*\). The nullspace of this linear transformation, that is equal to the set

\[
\mathfrak{F} = \{ F \in R_d \mid G(F) = 0 \text{ for all } G \in I_d \},
\]

has dimension 1. Let \( F \) be a nonzero polynomial in \( \mathfrak{F} \). Such a polynomial can be computed explicitly by using the matrix representation of \( T \) with respect to the basis \( \{f_1^*, \ldots, f_t^*\} \) for \((I_d)^*\) and the standard basis for \( R_d \). Indeed, this matrix is given by \((f_1^* \cdots f_t^*)^T \cdot (x_d^0 \cdots x_d^2)\). Here is an algorithm for finding \( F \):

\begin{verbatim}
Input: ideal I with Q=S/I artinian, arithmetically Gorenstein
Output: a nonzero polynomial F with I_S(F)=I
i:=0
r:=dim(Q_0)
d:=0
Repeat
  r=dim(Q_i)
  d=i-1
Until r=0
B:=a basis of I_d
B':=the standard basis for R_d
A:=B^T*B'
syz:=a syzygy matrix of A
F:=B'*syz
\end{verbatim}
In Macaulay2, we use the function \texttt{diff} to compute \( A \) in pseudocode. This function is used to differentiate polynomials. Basically, the first argument is the variable to differentiate with respect to, and the second one is the polynomial to be differentiated:

```macaulay2
i1 : R=QQ[x,y]
o1 = R
o1 : PolynomialRing

i2 : F=x^2*y+y^7
    o2 = y + x y
    o2 : R

i3 : diff(x,F)
    o3 = 2x*y
    o3 : R
```

The first argument can be also sum:

```macaulay2
i4 : diff(x+y,F)
    o4 = 7y + x + 2x*y
    o4 : R
```

The first and second arguments can be matrices:

```macaulay2
i5 : diff(transpose matrix{{x,y}},matrix{{x^3+y,x*y+y^2}})
    o5 = {1} | 3x2 y |
       {1} | 1 x+2y |
   2 2
    o5 : Matrix R <--- R
```

This corresponds to the jacobian matrix of the ideal generated by the matrix in the second argument.
Here is the function for finding $F$:

```plaintext
i6 : idealOfCurveCorrToGorenstein=(idl)->(
    i:=0;
    isMaximum:=false;
    r:=ring idl;
    numbasis:=numgens source basis(0,r/idl);
    maxi:=0;
    while not isMaximum do (
        numbasis=numgens source basis(i+1,r/idl);
        maxi=i;
        if numbasis===0 then ( 
            isMaximum=true;
            g:=(gens idl)* map(source gens idl,basis(maxi,idl));
            m:=basis(maxi,r);
            mat:=diff(transpose g,m);
            sy:=syz mat;
            f:=basis(maxi,r)*sy;
        );
        i=i+1;
    );
ideal f)
```

06 = idealOfCurveCorrToGorenstein

Problem 2 (Set 19). Let $J = (6xz - 5z^2, 6y^2 - 4z^2, 6xz - 3z^2, 6xy - 2z^2, 6x^2 - z^2)$. Then the quotient ring $Q$ of $S$ modulo $J$ is artinian and arithmetically Gorenstein. To check this, compute the free resolution of $Q$:

```plaintext
i7 : KK=QQ;
i8 : ringP2=KK[x,y,z];
i9 : J=ideal(6*y*z-5*z^2,6*y^2-4*z^2,6*x*z-3*z^2,6*x*y-2*z^2,6*x^2-z^2);
o9 : Ideal of ringP2
i10 : fJ=res J;
i11 : betti fJ
```

4
The free resolution of $Q$ is of length 4, and its last spot has rank 1. So $Q$ is an artinian and arithmetically Gorenstein ring. By using the function `idealOfCurveCorrToGorenstein`, we can compute the degree 2 polynomial $F$ in $R$ such that $J = I_S(F)$:

```plaintext
i12 : F = idealOfCurveCorrToGorenstein(J)

1 2 2 2 2 5 2
o12 = ideal(-x + -x*y + -y + x*z + -y*z + z)
6 3 3 3
```

```

o12 : Ideal of ringP2
```