Note. This script is also available at:

http://www.math.colostate.edu/~abo/Research/smi/smi-algebraic-geometry.html

1. Chow form of a line in $\mathbb{P}^3$

Let $k$ be an algebraically closed field, let $\mathbb{P}^3$ be the three-dimensional projective space over $k$. We denote by $S$ the homogeneous coordinate ring $k[x_0, x_1, x_2, x_3]$ of $\mathbb{P}^3$. Fix a line $L$ in $\mathbb{P}^3$. The general line in $\mathbb{P}^3$ does not intersect $L$. So the lines in $\mathbb{P}^3$ hitting $L$ form a proper subset $C(L)$ (actually a subvariety) of the grassmaniann of lines in $\mathbb{P}^3$. A question is: “How can we describe this subset?” Assume that the ideal $I(L)$ of $L$ is generated by the following two linear forms: $\sum_{i=0}^3 a_{i}x_i$ and $\sum_{i=0}^3 a_{i+1}x_i$. Let $L'$ be a line in $\mathbb{P}^3$ defined by linear forms $\sum_{i=0}^3 b_{i}x_i$ and $\sum_{i=0}^3 b_{i+1}x_i$. Then $L$ and $L'$ intersect if and only if the determinant of the matrix

$$\Lambda = \begin{pmatrix}
a_{00} & a_{01} & a_{02} & a_{03} \\
a_{10} & a_{11} & a_{12} & a_{13} \\
b_{00} & b_{01} & b_{02} & b_{03} \\
b_{10} & b_{11} & b_{12} & b_{13}
\end{pmatrix}$$

is zero. For $0 \leq i < j \leq 3$, let

$$\lambda_{ij} = a_{ij}a_{i+1,j+1} - a_{i+1,j}a_{ij+1}$$

and $\Lambda_{ij} = b_{ij}b_{i+1,j+1} - b_{i+1,j}b_{ij+1}$.

From Problem 3 in Problem Set 20, it follows that $\det(\Lambda) = 0$ if and only if

$$F = \lambda_{01}\Lambda_{23} - \lambda_{02}\Lambda_{13} + \lambda_{03}\Lambda_{12} + \lambda_{12}\Lambda_{03} - \lambda_{13}\Lambda_{02} + \lambda_{23}\Lambda_{01} = 0.$$

Recall that the Plücker embedding from $\mathbb{G}(1, 3)$ to $\mathbb{P}^5$ is defined by

$$L' \mapsto [\Lambda_{01} : \Lambda_{02} : \Lambda_{12} : \Lambda_{03} : \Lambda_{13} : \Lambda_{23}] = [X_0 : \cdots : X_5].$$

Consider the ring $k[b_{00}, \ldots, b_{13}, X_0, \ldots, X_5]$ and the ideal

$$I = (X_0 - \Lambda_{01}, X_1 - \Lambda_{02}, X_2 - \Lambda_{12}, X_3 - \Lambda_{03}, X_4 - \Lambda_{13}, X_5 - \Lambda_{23}, F).$$

Let $J = I \cap k[X_0, \ldots, X_5]$. Then $J = (Q, F')$, where

$$Q = X_0X_5 - X_1X_4 + X_2X_3$$

(1)
and
\[ F' = \lambda_{01}X_5 - \lambda_{02}X_4 + \lambda_{12}X_3 + \lambda_{03}X_2 - \lambda_{13}X_1 + \lambda_{23}X_0. \] (2)

Recall that \( Q \) is the defining equation of \( G(1,3) \). So \( C(L) \) can be regarded as a hypersurface in \( G(1,3) \). This hypersurface is called the **Chow variety** of \( L \), and the linear form \( F' \) is called the **Chow form** of \( L \). For a line \( L \) in \( \mathbb{P}^3 \) chosen at random, we compute the Chow form with Macaulay2:

```plaintext
i1 : KK=QQ;
i2 : ringP3=KK[x_0..x_3];
i3 : L=ideal random(ringP3^{0},ringP3^{2:-1})
      5 8 2 2
o3 = ideal (-*x + x , - -*x - -*x - -*x )
      2 1 2 9 0 5 1 3 2
o3 : Ideal of ringP3
i4 : coeff=transpose diff(transpose (vars ringP3),gens L)
      o4 = {-1} | 0 5/2 1 0 |
          | {-1} | -8/9 -2/5 -2/3 0 |
      2 4
o4 : Matrix ringP3 <--- ringP3
```

Using (2), we can compute the chow form of \( L \):
\[ F' = -\frac{19}{15}x_3 - \frac{8}{9}x_4 + \frac{20}{9}x_5. \]

Let’s check this!

```plaintext
i5 : ringP7=KK[b_(0,0)..b_(1,3)];
i6 : mat=matrix{{b_(0,0)..b_(0,3)},{b_(1,0)..b_(1,3)}}
o6 = | b_(0,0) b_(0,1) b_(0,2) b_(0,3) |
     | b_(1,0) b_(1,1) b_(1,2) b_(1,3) |
```

2
\( \begin{align*} 
&\text{o6 : Matrix ringP7} \quad \Leftarrow \quad \text{ringP7} \\
&\text{i7 : pluecker=\text{minors}(2,\text{mat})} \\
&\quad \text{o7 = ideal } (-b_{\ 0\ 1} b_{\ 1\ 0} - b_{\ 0\ 0} b_{\ 1\ 2}, -b_{\ 0\ 2} b_{\ 1\ 1} - b_{\ 0\ 1} b_{\ 1\ 2}, -b_{\ 0\ 3} b_{\ 1\ 1} - b_{\ 0\ 1} b_{\ 1\ 3}). \\
&\quad \text{o7 : Ideal of ringP7} \\
&\text{i8 : gamma=\text{substitute}(\text{coeff},\text{ringP7})||\text{mat}} \\
&\quad \text{o8 = \{\{-1\} | 0 \quad 5/2 \quad 1 \quad 0 \mid \\
&\quad \quad \{-1\} | -8/9 \quad -2/5 \quad -2/3 \quad 0 \mid \\
&\quad \quad \{0\} | b_{\(0,0\)} b_{\(0,1\)} b_{\(0,2\)} b_{\(0,3\)} \mid \\
&\quad \quad \{0\} | b_{\(1,0\)} b_{\(1,1\)} b_{\(1,2\)} b_{\(1,3\)} \mid} \\
&\quad \text{o8 : Matrix ringP7} \quad \Leftarrow \quad \text{ringP7} \\
&\text{i9 : F=\text{det gamma}} \\
&\quad \text{o9 = \frac{19}{15} b_{\ 0\ 3} b_{\ 1\ 0} + \frac{19}{9} b_{\ 0\ 3} b_{\ 1\ 1} - \frac{20}{9} b_{\ 0\ 3} b_{\ 1\ 2} - \frac{20}{9} b_{\ 0\ 3} b_{\ 1\ 3} - \frac{19}{15} b_{\ 0\ 3} b_{\ 1\ 0} + \frac{19}{9} b_{\ 0\ 3} b_{\ 1\ 1}} \\
&\quad \text{o9 : ringP7} \\
&\text{i10 : ringP5=KK[X_0..X_5];} \\
&\text{i11 : ringP7xP5=KK[b_{\(0,0\)}..b_{\(1,3\)},X_0..X_5,Degrees=>\{8:1,6:2\}, \\
&\quad \text{MonomialOrder=>Eliminate 8];} \\
&\text{i12 : grass=\text{substitute}(\text{vars ringP5},\text{ringP7xP5})- \\
&\quad \text{substitute}(\text{gens pluecker},\text{ringP7xP5});} \\
&\quad \text{o12 : Matrix ringP7xP5} \quad \Leftarrow \quad \text{ringP7xP5} \\
&\text{i13 : hyper=\text{substitute}(F,\text{ringP7xP5})} \\
\end{align*} \)
The quadratic polynomial \(Q\) in \(\text{chowVariety}\) looks a little different from (1). But this polynomial was just reduced. Indeed,

\[
Q = X_1X_4 + \frac{40}{57}X_2X_4 - X_0X_5 - \frac{100}{57}X_2X_5 \\
= X_1X_4 - X_0X_5 - X_2 \left( -\frac{40}{57}X_4 + \frac{100}{57}X_5 \right) \\
= X_1X_4 - X_0X_5 - X_2X_3.
\]

So \(Q\) differs from (1) by the sign.

The linear form we have obtained is equal to \(-\frac{15}{19}\). 

2. Number of 4-secant lines to four skew lines in \(\mathbb{P}^3\)

We start with the following question:
**Question 1.** Let $L_1, L_2, L_3$ and $L_4$ be four skew lines in $\mathbb{P}^3$. Is there a line which intersects all of them? If such a line exists, are there finitely many such lines or infinite many?

Suppose that there exists such a line $L$. Then $L$ can be regarded as a point in $\mathbb{G}(1,3)$. Since $L$ hits $L_1, L_2, L_3$ and $L_4$, the corresponding point in $\mathbb{G}(1,3)$ is contained in $C(L_1) \cap C(L_2) \cap C(L_3) \cap C(L_4)$. Let $F_i$ denote the Chow form of $L_i$, $i = 1, 2, 3, 4$. Then the intersection of the Chow varieties is defined by the ideal $I = (Q, F_1, F_2, F_3, F_4)$, where $Q$ is the defining equation of $\mathbb{G}(1,3)$ in $\mathbb{P}^5$. Since the ideal is generated by five polynomials, the corresponding variety $V(I)$ cannot be empty. From the generality of the choice of the four skew lines, we can expect that $\{Q, F_1, F_2, F_3, F_4\}$ is a minimal generating set for $I$. In this case, $\dim(V(I)) = 0$, that is, $V(I)$ is a finite set of points. The next question is therefore:

**Question 2.** How many points are there in $V(I)$?

Recall that the “degree” of a given hypersurface in $\mathbb{P}^n$ is defined to be the intersection number of the hypersurface itself and the general line in $\mathbb{P}^n$. On the other hand, the degree of an $r$-dimensional projective variety in $\mathbb{P}^n$ is defined to be $r!$ times the leading coefficient of its Hilbert polynomial (see Chapter I-7 in *Algebraic Geometry* by R. Hartshorne). The Hilbert polynomial $P_V$ of a hypersurface $V$ in $\mathbb{P}^n$ can be easily computed. Let $R$ be the homogeneous coordinate ring of $\mathbb{P}^n$. If the polynomial defining $V$ has degree $d$, then $P_V$ is obtained from the exact sequence:

$$0 \to R(-d) \to R \to \Gamma(V) \to 0.$$

Indeed, we obtain

$$P_V(t) = \binom{n+t}{n} - \binom{n+t-d}{n} = \frac{d}{(n-1)!}t^{n-1} + \cdots.$$ 

So $\deg(V) = d$. This implies that $\deg(\mathbb{G}(1,3)) = 2$, because $\deg(Q) = 2$. The Chow forms $F_1, F_2, F_3$ and $F_4$ define a line in $\mathbb{P}^5$, and this line meets $V(Q)$ exactly in two points, because otherwise the line would lie on $V(Q)$ and there are infinitely many lines which intersect all four lines. But this contradicts our assumption. Therefore the number of points in $V(I)$ is expected to be 2.

**Exercise (Problem 1 in Problem Set 21).** Given four skew lines in $\mathbb{P}^3$, show that the number of lines which intersect all of them is equal to 2.

**Hint.** Use either Formula (2) or the Macaulay2 script to get the Chow forms of the four lines.