Sample Exam 2 (Math461 Fall 2009)

Problem 1 (Binary operations)

Determine whether the given set is closed under the binary operation $*$ defined on $\mathbb{Z}$. If B is not closed, echibit elements $x \in B$ and $y \in B$ such that $x * y \notin B$.

(i) $x * y = |x| - |y|$, $B = \mathbb{N}$.

(ii) $x * y = \operatorname{sgn}(x) + \operatorname{sgn}(y)$, $B = \{-2, -1, 0, 1, 2\}$, where

$$\operatorname{sgn}(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0. \end{cases}$$

Problem 2 (Binary operations)

In each part following, a rule is given that determines a binary operation $*$ on $\mathbb{Z}$. Determine in each case whether the operation is commutative or associative and whether there is an identity element. Also find the inverse of each invertible element.

(i) $x * y = 3(x + y)$.

(ii) $x * y = x + 2y$.

(iii) $x * y = x + xy + y - 2$.

(iv) $x * y = |x| - |y|$.

Problem 3 (Subgroups)

Let $H = \{\ a + bi \mid a, b \in \mathbb{R}, a^2 + b^2 = 1\ \} \subset \mathbb{C}^\times$, where i is a root of $x^2 + 1$. Prove or disprove that H is a subgroup of $\mathbb{C}^\times$.

Problem 4 (Subgroups)

Let $H = \{A \in GL(2, \mathbb{R}) \mid \det(A) \text{ is a power of } 2\ \}$. Show that H is a subgroup of $GL(2, \mathbb{R})$.

Problem 5 (Subgroups)

Let G be an abelian group with the binary operation written as multiplication. For a fixed positive integer n, let

$$G_n = \{a \in G \mid a = x^n \text{ for some } x \in G\}.$$

Prove that G_n is a subgroup of G.

Problem 6 (Center)

Let G be a group and let $Z(G) = \{a \in G \mid ax = xa \text{ for all } x \in G\}$. Show that

$$Z(G) = \bigcap_{a \in G} C_a(G),$$

where $C_a(G) = \{x \in G \mid ax = xa\}$ is the centralizer of a in G.

Note. The subset $Z(G)$ of G is a subgroup. The proof can be found in the textbook, pg 65.

Problem 7 (Cyclic groups)

Let $U(\mathbb{Z}_{11})$ be the set of nonzero elements of $\mathbb{Z}_{11}$.

(i) Verify that $U(\mathbb{Z}_{11})$ is generated by [2].

(ii) Find the other generators of $U(\mathbb{Z}_{11})$.

(iii) Determine the number of distinct subgroups of $U(\mathbb{Z}_{11})$.

Problem 8 (Order)

Let G be a group with the binary operation written as multiplication. Let a and b be elements of G of order m and n respectively. Suppose that $ab = ba$. Prove that if $\langle a \rangle \cap \langle b \rangle = \{e\}$, then G contains an element whose order is the least common multiple of m and n.

Note. The *least common multiple* of two non-zero integers a and b, denoted $\operatorname{lcm}(a, b)$, is the smallest positive integer that is a multiple of both a and b.

Problem 9 (Order)

Let G be a cyclic group of order 24. Let $a \in G$. Prove that if $a^8 \neq e$ and $a^{12} \neq e$, then $G = \langle a \rangle$.