Homework 2 (Math462)

Problem 1 (4 points)

Let $f(x) = x^2 + 1 \in \mathbb{R}[x]$ and let $I = \{s(x)f(x) \mid s(x) \in \mathbb{R}[x]\}$.

(i) Prove that $\mathbb{R}[x]/I$ is a field, i.e., every non-zero element of $\mathbb{R}[x]/I$ has an inverse.

(ii) Recall that $\mathbb{R}[x]/I$ is a vector space over $\mathbb{R}$. Find the dimension of $\mathbb{R}[x]/I$.

Problem 2 (4 points)

Let $f(x) = x^2 - 2 \in \mathbb{Q}[x]$ and let $I = \{s(x)f(x) \mid s(x) \in \mathbb{Q}[x]\}$. Then

$$\mathbb{Q}[x]/I = \{(ax + b) + I \mid a, b \in \mathbb{Q}\}.$$

Let $\mathbb{Q}[\sqrt{2}] = \{a\sqrt{2} + b \mid a, b \in \mathbb{Q}\}$ and let $\varphi : \mathbb{Q}[x]/I \to \mathbb{Q}[\sqrt{2}]$ be the map defined by $\varphi((ax + b) + I) = \sqrt{2}a + b$.

(i) Prove that φ is well-defined.

(ii) Prove that φ is a group homomorphism, i.e., $\varphi([g(x)+I]+[h(x)+I]) = \varphi(g(x) + I) + \varphi(h(x) + I)$.

(iii) Prove that $\varphi([g(x) + I][h(x) + I]) = \varphi(g(x) + I)\varphi(h(x) + I)$.

(iv) Prove that φ is an isomorphism.

Problem 3 (2 points)

Let $f(x) = x^3 + 2x$, $g(x) = x^2 + x + 1$, $h(x) = 2x^3 + x^2 + 2x + 1 \in \mathbb{Z}_3[x]$.

(i) Prove that $f(x)$, $g(x)$ and $h(x)$ are linearly dependent.

(ii) Let $I = \{s_1(x)f(x)+s_2(x)g(x)+s_3(x)h(x) \mid s_1(x), s_2(x), s_3(x) \in \mathbb{Z}_3[x]\}$. Find the dimension of $\mathbb{Z}_3[x]/I$.