Homework 4 (Math462)

Problem 1 (2.5 points)

Let $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{(0,0),(1,1),(0,1),(1,0)\}$. Addition and multiplication are defined on $\mathbb{Z}_2 \times \mathbb{Z}_2$:

- $(a,b)+(c,d) = (a+c, b+d)$.
- $(a,b)(c,d) = (ad+bc+bd, ad+bc+ac)$.

(i) Prove that multiplication is associative.

Assume that S is a ring with respective the binary operations as defined above.

(ii) Is S a commutative ring?

(iii) Does S have a unity?

(iv) Is S an integral domain?

(v) Is S a field?

Problem 2 (2.5 points)

Prove that if a subring R of an integral domain D contains the unity of D, then R is an integral domain.

Problem 3 (2.5 points)

The subset $S = \left\{ \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \,\middle|\, x,y,z \in \mathbb{Z} \right\}$ of $M_{2\times 2}(\mathbb{Z})$ is a subring. Show that

$$I = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \,\middle|\, a,b \in \mathbb{Z} \right\}$$

is an ideal of S.

Problem 4 (2.5 points)

Let I_1 and I_2 be ideals of a ring R. Prove that $I_1 + I_2 = \{x+y \mid x \in I_1, y \in I_2\}$ is an ideal of R that contains both I_1 and I_2.