Homework 5 (Math462)

Problem 1 (4 points)

(i) Let R be a ring with unity $1 \neq 0$. Show that if an ideal I of R contains a unit, then $I = R$.

(ii) Let F be a field. Then the only ideals of F are $\{0\}$ and F itself.

Problem 2 (3 points)

Let $I = \{0, 3\} \subset \mathbb{Z}_6$. Determine whether or not I is a prime ideal. Justify your answer.

Problem 3 (3 points)

Let $I = \{(3x, y) \mid x, y \in \mathbb{Z}\} \subset \mathbb{Z} \times \mathbb{Z}$. Prove that I is a maximal ideal.

Note. Recall that addition and multiplication are defined on $\mathbb{Z} \times \mathbb{Z}$:

- $(a, b) + (c, d) = (a + c, b + d)$.
- $(a, b)(c, d) = (ac, bd)$.

Bonus Problem (2 points)

Let R be a commutative ring with unity that has the property that $a^2 = a$ for all $a \in R$ and let I be a prime ideal of R. Show that $|R/I| = 2$.