Homework 9 (Math462)

Problem 1 (3 points)

Prove that $2 \pm \sqrt{-5}$ are irreducible in $\mathbb{Z}[\sqrt{-5}]$.

Problem 2 (4 points)

Let R be a ring with unity 1_R.

(i) Prove that $\text{char}(R) = 0$ if and only $\not\exists n \in \mathbb{N}$ such that $n \cdot 1_R = \underbrace{1_R + \cdots + 1_R}_{n \text{ times}} = 0$.

(ii) Prove that $\text{char}(R) = n > 0$ if and only n is the least positive integer such that $n \cdot 1_R = 0$.

Problem 3 (3 points)

Let E be a field and let p be a prime number. Prove that if $\text{char}(E) = p$, then E is an extension field of $\mathbb{Z}_p$.

Hint. Define $\varphi : \mathbb{Z} \to E$ by $\varphi = n \cdot 1_E$ for $\forall n \in \mathbb{Z}$, describe $\ker(\varphi)$ and then use the first isomorphism theorem to prove that E contains the subring that is isomorphic to $\mathbb{Z}_p$.