Sample Exam 1 (Math461 Fall 2009)

Problem 1 (Sets and Maps)

(i) Let A and B be sets. Prove that $A \cup B = (A \setminus B) \cup B$.

(ii) Let A and B be sets and let $f : A \to B$ be a map. Prove that if A_1 is a subset of A, then $A_1 \subseteq f^{-1}(f(A_1))$.

Remark.

- $A \setminus B = \{x \in A | x \notin B\}$.
- Let $f : A \to B$ be a map and let B_1 be a subset of B. Then $f^{-1}(B_1) = \{x \in A | f(x) \in B_1\}$.

Problem 2 (Maps)

Let $\mathbb{R}^\times = \mathbb{R} \setminus \{0\}$ be the set of non-zero elements in $\mathbb{R}$ and let $f : \mathbb{R}^\times \to \mathbb{R}$ be the map defined by

$$f(x) = \frac{x-1}{x}.$$

Determine whether f is one-to-one and whether it is onto. Justify your answer.

Problem 3 (Binary Relation)

Let $\mathbb{R}^\times = \mathbb{R} \setminus \{0\}$. Consider the following relations on $\mathbb{R}^\times \times \mathbb{R}^\times$. Determine in each case whether R is an equivalence relation, and justify your answer.

(i) $(a, b)R(c, d)$ if and only if $ad = bc$.

(ii) $(a, b)R(c, d)$ if and only if $a - b = c - d$.

Problem 4 (Equivalence Classes)

Let R be the relation on $\mathbb{Z}$ defined by xRy if and only if $x + 3y$ is a multiple of 4. Prove that R is an equivalence relation. Find the distinct equivalence classes of R and list at least four members of each.

Problem 5 (Mathematical Induction)

Use mathematical induction to prove that the following statement is true for every positive integer n

$$\frac{1}{1\cdot 2\cdot 3}+\frac{1}{2\cdot 3\cdot 4}+\cdots+\frac{1}{n(n+1)(n+2)}=\frac{n(n+3)}{4(n+1)(n+2)}.$$

Problem 6 (Mathematical Induction)

Use mathematical induction to prove that 8 divides 9^n-1 for all positive integer n.

Problem 7 (Euclidean Algorithm)

Let $a=5088$ and let $b=156$. Use the Euclidean Algorithm to find $d=\gcd(a,b)$ and integers s and t such that $d=as+bt$.

Problem 8 (Binary Operation)

Find the multiplicative inverse of $[33]\in\mathbb{Z}_{58}$.

Problem 9 (Divisors)

Prove that if n is a positive integer greater than 1 such that n is not a prime, then n has a divisor d such that $1<d\leq\sqrt{n}$.

Problem 10 (Prime Numbers)

Let n and r be integers with $1\leq r\leq n-1$. The binomial coefficient $\binom{n}{r}$ is defined by

$$\binom{n}{r}=\frac{n!}{(n-r)!r!}.$$

Prove that if p is a prime, then $\binom{p}{r}$ is divisible by p.

Hint. You may assume that $\binom{n}{r}$ is an integer.