Homework 1 (Math462)

Problem 1 (2 points)

For $f(x)$, $g(x) \in \mathbb{Z}_p[x]$ given in (i) and (ii), find the quotient and remainder when dividing $f(x)$ by $g(x)$ in $\mathbb{Z}_p[x]$.

(i) $f(x) = [2]x^3 + [3]x^2 + [4]x + [1]$, $g(x) = [3]x + [1] \in \mathbb{Z}_5[x]$.

(ii) $f(x) = [1]x^4 + [2]x^2 + [1]x + [1]$, $g(x) = [1]x^3 + [1]x^2 + [2]x + [2] \in \mathbb{Z}_3[x]$.

Problem 2 (2 points)

For $f(x)$, $g(x) \in \mathbb{Z}_p[x]$ given in (i) and (ii), find $\gcd(f(x), g(x))$.

(i) $f(x) = [1]x^4 + [2]x^2 + [1]x + [1]$, $g(x) = [1]x^3 + [1]x^2 + [2]x + [2] \in \mathbb{Z}_3[x]$.

(ii) $f(x) = [1]x^4 + [5]x^2 + [2]x + [2]$, $g(x) = [3]x^2 + [2] \in \mathbb{Z}_7[x]$.

Problem 3 (2 points)

Let $f(x) = x^2 + 1 \in \mathbb{Z}_2[x]$ and let $I = \{s(x)f(x) \mid s(x) \in \mathbb{Z}_2[x]\}$. Recall that multiplication on $\mathbb{Z}_2[x]/I$ is defined by $[g(x) + I][h(x) + I] = g(x)h(x) + I$. Complete the multiplication table for $\mathbb{Z}_2[x]/I$.

Note. $\mathbb{Z}_2[x]/I$ has exactly four elements, namely I, $1+I$, $x+I$ and $(x+1)+I$.

Problem 4 (2 points)

Let $R = \mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{Z}_\ell$, let $f(x) \in R[x]$ and let $I = \{s(x)f(x) \mid s(x) \in R[x]\}$. Prove that if $g(x)$, $h(x)$, $k(x) \in R[x]$, then

$$[g(x)+I]\left\{[h(x) + I] + [k(x) + I]\right\} = [g(x)+I][h(x)+I]+[g(x)+I][k(x)+I].$$

Problem 5 (2 points)

Let $F = \mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{Z}_p$, where p is prime, let $f(x)$, $g(x) \in F[x]$ and let $I = \{s(x)f(x) + t(x)g(x) \mid s(x), t(x) \in F[x]\}$.

(i) Prove that I is a subgroup of $F[x]$.

(ii) Suppose that one of $f(x)$ and $g(x)$ is non-zero. Let $d(x) = \gcd(f(x), g(x))$. Prove that $I = \{u(x)d(x) \mid u(x) \in F[x]\}$.