Sample Exam 2 (Math462 Spring 2010)

Problem 1 (Ideals)

Let I be an ideal of a ring R. Prove that

$$J = \{r \in R \mid ra = 0 \text{ for all } a \in I\}$$

is an ideal of R.

Problem 2 (Ideals)

Given that the set

$$S = \left\{ \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \;\middle|\; x, y, z \in \mathbb{Z} \right\}$$

is a ring with respect to matrix addition and multiplication, show that

$$I = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \;\middle|\; a, b \in \mathbb{Z} \right\}$$

is an ideal of S.

Problem 3 (Prime ideals)

Prove or disprove that the ideal (2) of $\mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} \mid a, b \in \mathbb{Z}\}$ generated by 2 is a prime ideal $\mathbb{Z}[\sqrt{-5}]$.

Problem 4 (Maximal ideals)

Let $\mathbb{E} = \{2n \mid n \in \mathbb{Z}\}$. Show that the ideal (6) of $\mathbb{E}$ generated by 6 is a maximal ideal.

Problem 5 (Maximal ideals)

Let R be a principal ideal domain. Prove that every non-trivial prime ideal I of R (i.e., $I \neq \{0\}$) is a maximal ideal.

Problem 6 (Ring homomorphisms)

Each of the following rules determines a map $\phi : \mathbb{R} \to \mathbb{R}$. Decide in each case whether ϕ is a ring homomorphism.

(i) $\phi(x) = |x|$.

(ii) $\phi(x) = 2x$.

(iii) $\phi(x) = x^2$.

Problem 7 (Ring homomorphisms)

Let R and S be commutative rings, let I be an ideal of S, let $f : R \to S$ be a ring homomorphism and let $f^{-1}(I) = \{x \in R \mid f(x) \in I\}$. Prove that if I is a prime ideal, then so is $f^{-1}(I)$.

Problem 8 (Mixed)

Make each of the following true or false.

(i) A ring homomorphism $\varphi : R \to R'$ carries ideals of R into R'.

(ii) A ring homomorphism is one to one if and only if its kernel is $\{0\}$.

(iii) A proper subset I of a ring commutative ring R with unity $1 \neq 0$ is a prime ideal if and only if I is a maximal ideal.

Problem 9 (Homomorphisms)

Consider the map $\varphi : \mathbb{Z}_{12} \to \mathbb{Z}_{12}$ defined by $\varphi([a]) = 4[a]$. Determine whether or not φ is a ring homomorphism. Justify your answer.

Problem 10 (Isomorphism Theorem)

Let I and J be ideals of a ring R such that $I \subseteq J$.

(i) Show that J/I is an ideal of R/I.

(ii) R/J is isomorphic to the factor ring $(R/I)/(J/I)$ of R/I modulo J/I.