Adams-Bashforth 2-step Method

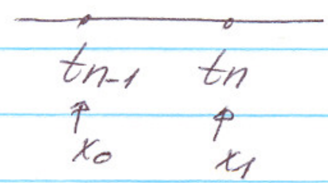
$$y'(t) = f(t, y(t)) \qquad a \le t \le b$$

$$y(a) = \alpha$$

Integrate $\displaystyle\int_{t_n}^{t_{n+1}}$

$$\int_{t_n}^{t_{n+1}} y'(t)\,dt = \int_{t_n}^{t_{n+1}} f(t, y(t))\,dt$$

$$y(t_{n+1}) - y(t_n) = \int_{t_n}^{t_{n+1}} f(t, y(t))\,dt$$



$$f(t, y(t)) = p_1(t) + E_1(t)$$

$$p_1(t) = l_0(t)\cdot f(t_{n-1}, y_{n-1}) + l_1(t)\cdot f(t_n, y_n)$$

$$E_1(t) = \frac{\partial^2 f}{\partial t^2}(\xi, y(\xi))\cdot(t - t_{n-1})(t - t_n)$$

$$l_0(t) = \frac{t - t_n}{t_{n-1} - t_n} \qquad l_1(t) = \frac{t - t_{n-1}}{t_n - t_{n-1}}$$

$$\int_{t_n}^{t_{n+1}} f(t, y(t))\,dt = \int_{t_n}^{t_{n+1}} p_1(t)\,dt + \int_{t_n}^{t_{n+1}} E_1(t)$$

$$\int_{t_n}^{t_{n+1}} p_1(t)\,dt = f(t_{n-1}, y_{n-1})\int_{t_n}^{t_{n+1}} l_0(t)\,dt + f(t_n, y_n)\int_{t_n}^{t_{n+1}} l_1(t)\,dt$$

$$\int_{t_n}^{t_{n+1}} l_0(t)\,dt = \frac{1}{t_{n-1}-t_n}\int_{t_n}^{t_{n+1}} (t-t_n)\,dt = \left|\begin{array}{c} t = t_n + sh \\ t - t_n = sh \\ dt = h\,ds \end{array}\right| =$$

$$= \frac{1}{t_{n-1}-t_n}\int_0^1 sh\cdot h\,ds = \frac{h^2}{-h}\; s^2\Big|_0^1 = -\frac{h}{2}$$

$$\int_{t_n}^{t_{n+1}} l_1(t)\,dt = \frac{1}{t_n - t_{n-1}}\int_{t_n}^{t_{n+1}} (t-t_{n-1})\,dt = \left|\begin{array}{c} t = t_n + sh \\ t - t_{n-1} = t_n - t_{n-1} + sh \\ = h(s+1) \end{array}\right| =$$

$$= \frac{1}{h}\cdot\int_0^1 h(s+1)\cdot h\,ds = \frac{h^2}{h}\left(\frac{s^2}{2}+s\right)\Big|_0^1 = h\cdot\frac{3}{2} = \frac{3h}{2}$$

$$\Rightarrow \int_{t_n}^{t_{n+1}} p_1(t)\,dt = -\frac{h}{2}\, f(t_{n-1}, y_{n-1}) + \frac{3h}{2}\, f(t_n, y_n)$$

$$\int_{t_n}^{t_{n+1}} E_1(t)\,dt = \int_{t_n}^{t_{n+1}} \frac{1}{2}\frac{\partial^2 f}{\partial t^2}\left(\xi(t), y(\xi(t))\right)\cdot(t-t_{n-1})(t-t_n)\,dt =$$

$$\underset{MVT}{\overset{\text{generalized}}{=}} \frac{1}{2}\frac{\partial^2 f}{\partial t}\left(\hat{\xi}, y(\hat{\xi})\right)\cdot\int_{t_n}^{t_{n+1}} (t-t_{n-1})(t-t_n)\,dt \;=$$

$$t = t_n + sh$$

$$t - t_n = sh; \quad t - t_{n-1} = t_n - t_{n-1} + sh = (1+s)h$$

$$= \frac{1}{2}\frac{\partial^2 f}{\partial t^2}\left(\hat{\xi}, y(\hat{\xi})\right)\cdot\int_0^1 sh\cdot(1+s)h\cdot h\,ds = \frac{1}{2}\frac{\partial^2 f}{\partial t^2}\left(\hat{\xi}, y(\hat{\xi})\right)\cdot h^3\int_0^1 \underbrace{s(1+s)}_{s+s^2}\,ds$$

$$= \underbrace{\frac{1}{2}\frac{\partial^2 f}{\partial t^2}\left(\hat{\xi}, y(\hat{\xi})\right)}_{y'''(\hat{\xi})} h^3\left(\frac{s^2}{2}+\frac{s^3}{3}\right)\Big|_0^1 = \frac{5}{12}\, h^3\, y'''(\hat{\xi})$$

Hence,

$$y_{n+1} - y_n = \frac{h}{2}\left(3f(t_n, y_n) - f(t_{n-1}, y_{n-1})\right) + \underbrace{\frac{5}{12}h^3 y'''(\xi)}_{\tau_n}$$

then

$$u_{n+1} - u_n = \frac{h}{2}\left(3f(t_n, u_n) - f(t_{n-1}, u_{n-1})\right)$$

$$\tau_n = \frac{5}{12}h^3 y'''(\xi) = O(h^3) \Rightarrow 2^{nd} \text{ order accurate method}$$