MATH 471: Introduction to Analysis I

HW #3: SOLUTIONS

Section 2.6

#1

(a) $b_n = -1$ and $a_n = (-1)^n$

Yes, $\{b_n\}$ is a subsequence of a_n with $n = 2k-1$: odd

$$b_n = a_{2n-1}$$

(b) $b_n = \frac{1}{\sqrt{n}}$ and $a_n = \frac{1}{n}$

No, $\{b_n\}$ is not a subseq. of $\{a_n\}$ since, for example,

$b_2 = \frac{1}{\sqrt{2}}$ is not a term of $\{a_n\}$

(c) $b_n = \frac{1}{3n^2}$ and $a_n = \frac{1}{n^2}$

No, $\{b_n\}$ is not a subseq. of $\{a_n\}$ since, for instance,

$b_1 = \frac{1}{3}$ is not a term of $\{a_n\}$.

#2

(a) $a_n = \dfrac{1+(-1)^{n+1}}{2}$ $\quad a_{2k} = 0, \quad a_{2k-1} = 1$

$\lim_{k\to\infty} a_{2k} = 0$, $\lim_{k\to\infty} a_{2k-1} = 1$. Limits are different

$\Rightarrow \{a_n\}$ diverges.

Limit points: 0 and 1

$\limsup_{n\to\infty} a_n = \overline{\lim_{n\to\infty}} a_n = 1$; $\liminf_{n\to\infty} = \underline{\lim_{n\to\infty}}\, a_n = 0$

(b) $a_n = \sin \frac{n\pi}{2}$ sequence $\{a_n\}$ is $1, 0, -1, 0, 1, \ldots$

$a_{2k} = 0 \quad \lim_{k\to\infty} a_{2k} = 0$

$\{a_{2k-1}\}$ is $1, -1, 1, -1, \ldots \quad a_{4k-3} = 1, \quad a_{4k-1} = -1$

$\lim_{k\to\infty} a_{4k-3} = 1, \quad \lim_{k\to\infty} a_{4k-1} = -1$

limits are different $\Rightarrow$ $\{a_n\}$ diverges

limit points: $-1, 0, 1$

$\overline{\lim_{n\to\infty}}\, a_n = 1, \quad \underline{\lim_{n\to\infty}}\, a_n = -1$

(c) $a_n = r^n, \; r \le -1$ or $r > 1$

– if $r = -1 \Rightarrow a_n = (-1)^n \quad a_{2k} = 1, \; a_{2k-1} = -1$

$\lim_{k\to\infty} a_{2k} = 1, \; \lim_{k\to\infty} a_{2k-1} = -1$. limits are different.

$\Rightarrow$ $\{a_n\}$ diverges

limit points are ± 1. $\overline{\lim_{n\to\infty}}\, a_n = 1, \; \underline{\lim_{n\to\infty}}\, a_n = -1$

– if $r > 1$ then $\lim_{n\to\infty} r^n = +\infty \Rightarrow$ no limit point

– if $r < -1$ then $\lim_{n\to\infty} r^n$ does not exist $\Rightarrow$ no limit pt

(d) $a_n = (-1)^n \frac{n-1}{n} \quad \lim_{n\to\infty} a_{2n} = 1, \; \lim_{n\to\infty} a_{2n-1} = -1$

limits are different $\Rightarrow$ $\{a_n\}$ diverge

limit points: ± 1

$\underline{\lim_{n\to\infty}}\, a_n = -1, \; \overline{\lim_{n\to\infty}}\, a_n = 1$

Section 3.1 #10

(a) For $\forall x_{\geq 1}\ \exists n\in N: \quad n \leq x \leq n+1 \Rightarrow$

$$\Rightarrow \frac{1}{n+1} \leq \frac{1}{x} \leq \frac{1}{n} \Rightarrow 1+\frac{1}{n+1} \leq 1+\frac{1}{x} \leq 1+\frac{1}{n}$$

$$\Rightarrow \left(1+\frac{1}{n+1}\right)^n \overset{①}{\leq} \left(1+\frac{1}{x}\right)^x \leq \left(1+\frac{1}{n}\right)^{n+1}$$

$$\left(1+\frac{1}{n+1}\right)^n = \left(1+\frac{1}{n+1}\right)^{n+1-1} = \underbrace{\left(1+\frac{1}{n+1}\right)^{n+1}}_{\downarrow\ e \text{ as } n\to\infty} \cdot \frac{1}{\underbrace{1+\frac{1}{n+1}}_{\downarrow\ 1 \text{ as } n\to\infty}} \to e \text{ as } n\to\infty$$

$$\left(1+\frac{1}{n}\right)^{n+1} = \underbrace{\left(1+\frac{1}{n}\right)^n}_{\downarrow\ e \text{ as } n\to\infty} \cdot \underbrace{\left(1+\frac{1}{n}\right)}_{\downarrow\ 1 \text{ as } n\to\infty} \to e \text{ as } n\to\infty$$

Hence, by Sandwich Thm $\left(1+\frac{1}{x}\right)^x \to e$ as $x\to\infty$

(Note: as $n\to\infty \Rightarrow x\to\infty$ as well).

(b) $$\lim_{x\to\infty}\left(\frac{x}{x+1}\right)^x = \frac{1}{\lim_{x\to\infty}\left(\frac{x+1}{x}\right)^x} = \frac{1}{\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x} \overset{\text{part (a)}}{=} \frac{1}{e}$$

(c) $$\lim_{x\to\infty}\left(1+\frac{1}{\sqrt{x}}\right)^x = \left| t=\sqrt{x} \right| = \lim_{t\to\infty}\left(1+\frac{1}{t}\right)^{t^2}$$

$$\left(1+\frac{1}{t}\right)^{t^2} = \left(1+\frac{1}{t}\right)^{t\cdot t} = \left(\left(1+\frac{1}{t}\right)^t\right)^t \overset{①}{\geq} \left[\left(1+\frac{1}{n+1}\right)^n\right]^t$$

$$\left(1+\frac{1}{n+1}\right)^n \to e \text{ as } n\to\infty \Rightarrow \left[\left(1+\frac{1}{n+1}\right)^n\right]^t \to e^t \text{ as } n\to\infty$$

$e^t \to \infty$ as $t\to\infty$

By comparison test $\left(\left(1+\frac{1}{t}\right)^t\right)^t \to \infty$ as $t \to \infty$

#12 $f: (a,\infty) \to \mathbb{R}$, $g: (a,\infty) \to \mathbb{R}$, $a \in \mathbb{R}$

$\lim_{x\to\infty} f(x) = L$, $\lim_{x\to\infty} g(x) = +\infty \overset{?}{\Rightarrow} \lim_{x\to\infty} (f\circ g)(x) = L$

Proof

(2) $\lim_{t\to\infty} f(t) = L \Rightarrow \forall \varepsilon > 0 \ \exists M > 0: \forall t > M \Rightarrow |f(t) - L| < \varepsilon$

(3) $\lim_{x\to\infty} g(x) = +\infty \Rightarrow \forall K > 0 \ \exists P > 0: \forall x > P \Rightarrow g(x) > K$

Consider $f\circ g$. Need to show:

$\forall \varepsilon > 0 \ \exists R > 0: x > R \Rightarrow |(f\circ g)(x) - L| < \varepsilon$

$|(f\circ g)(x) - L| = |f(g(x)) - L| = |t = g(x)| = |f(t) - L|$

from (2) Given $\varepsilon > 0$, we can find $M > 0$: $\forall t > M: |f(t) - L| < \varepsilon$

from (3) When $t = g(x) > M$, we can find $P > 0$: $\forall x > P \ \ g(x) > M$

Hence, given $\forall \varepsilon > 0$, we can find $R = P$: $\forall x > R$

$g(x) > M$ and $|f(g(x)) - L| < \varepsilon$

$\Rightarrow f\circ g \to L$ as $x \to \infty$

Section 3.2

#14

(a) $\lim_{x\to 3a} f(x) = 3\lim_{x\to a} f(x)$ False

Ex $f(x) = x^2 \Rightarrow \lim_{x\to 3a} x^2 = (3a)^2 = 9a^2$

$\#$

$3\lim_{x\to a} x^2 = 3a^2$

(b) $\lim_{x\to a} f(3x) = 3\lim_{x\to a} f(x)$ False

Ex $f(x) = x^2 \Rightarrow \lim_{x\to a} 9x^2 = 9a^2$

$f(3x) = (3x)^2 = 9x^2$

$\neq$

$3\lim_{x\to a} f(x) = 3\lim_{x\to a} x^2 = 3a^2$

(c) $\lim_{x\to 3a} f(x) = \lim_{t\to a} f(3t)$ true

$$\lim_{t\to a} f(3t) = \left| \begin{array}{l} x = 3t \\ t\to a \Rightarrow x\to 3a \end{array} \right| = \lim_{x\to 3a} f(x) \quad \checkmark$$

By def:

(4) $\lim_{x\to 3a} f(x) = L \Rightarrow \forall \varepsilon>0\ \exists \delta>0: 0<|x-3a|<\delta \Rightarrow |f(x)-L|<\varepsilon$

$\lim_{t\to a} f(3t) = L \Rightarrow \forall \varepsilon>0\ \exists \delta_1>0: 0<|t-a|<\delta_1 \Rightarrow |f(3t)-L|<\varepsilon$

let $x=3t \Rightarrow \exists \delta_1>0: 0<|\frac{x}{3}-a|<\delta_1 \Rightarrow |f(x)-L|<\varepsilon$

$0<|\frac{x}{3}-a|<\delta_1 \Leftrightarrow 0<|x-3a|<3\delta_1$: same as (4) with $\delta = 3\delta_1$