Math 471: Introduction to Analysis

Homework #4. SOLUTIONS

Section 33 $\quad f: \mathbb{R} \to \mathbb{R}$

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(a) $\lim_{x\to 0} f(x) = L$ implies that $\lim_{x\to\infty} f(\frac{1}{x}) = L$

(b) $\lim_{x\to\infty} f(\frac{1}{x}) = L$ implies that $\lim_{x\to 0} f(x) = L$

Solution (a) $\lim_{x\to\infty} f(\frac{1}{x}) = \lim_{x\to+\infty} f(\frac{1}{x}) = \left| \begin{array}{l} t = \frac{1}{x} \\ x\to+\infty \Rightarrow t\to 0^+ \end{array} \right| =$

$= \lim_{t\to 0^+} f(t) \;(=)$

since $\lim_{x\to 0} f(x) = L \Rightarrow \lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = L$ and $\exists$ finite

$\Rightarrow (=)\; L \quad \Rightarrow$ (a) is true

(b) $\lim_{x\to\infty} f(\frac{1}{x}) = L \Leftrightarrow \lim_{x\to+\infty} f(\frac{1}{x}) = L = \left| \begin{array}{l} t = \frac{1}{x} \\ x\to+\infty \Rightarrow t\to 0^+ \end{array} \right|$

$= \lim_{t\to 0^+} f(t) = L$

Note $\lim_{t\to 0^-} f(t)$ may not exist or if it exists, it may not be equal to L

If $\lim_{x\to 0} f(x)$ exists and equals $L \Rightarrow \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = L$

but $\lim_{x\to 0^-} f(x)$ may not exist or may not $= L$ as argued above $\Rightarrow$ (b) is false

Couter Example for (b): $f(x)=\sqrt{x}$

$\lim_{x\to 0+} \sqrt{x}=0$ while $\lim_{x\to 0-} \sqrt{x}$ doesn't exist since $\sqrt{x}$ is not defined for $x<0$.

Section 4.1

(#5) $f, g : D \to \mathbb{R}$ are continuous at $x=a$, then the given functions are also continuous at $x=a$

(a) $|f|$

(b) $\sqrt{f}$, $f(x)\geq 0$ $\forall x\in D$

(c) $\max\{f, g\}$

(d) $\min\{f, g\}$

(e) $[f(x)]^n$, $n\in \mathbb{N}$

Solution (a) Show that $\lim_{x\to a} |x| = |a|$

$\forall \varepsilon>0$ $\exists \delta>0 : 0<|x-a|<\delta \Rightarrow \big||x|-|a|\big|<\varepsilon$

$\big||x|-|a|\big| \leq |x-a| < \delta \Rightarrow \delta=\varepsilon$

$\Rightarrow g(x)=|x|$ is continuous at $x=a$

Then $|f| = |x| \circ f$ is also continuous as a composition of continuous functions by Thm 4.1.9

(b) Similar to part (a)

Show that $\lim_{x\to a}\sqrt{x}=\sqrt{a}$

$\forall \varepsilon > 0 \quad \exists \delta > 0: \quad |x-a| < \delta \Rightarrow \left|\sqrt{x}-\sqrt{a}\right| < \varepsilon$

$$\left|\sqrt{x}-\sqrt{a}\right| = \left|\frac{(\sqrt{x}-\sqrt{a})(\sqrt{x}+\sqrt{a})}{\sqrt{x}+\sqrt{a}}\right| = \left|\frac{x-a}{\sqrt{x}+\sqrt{a}}\right| \; < $$

$$|x-a| < \delta \Rightarrow \quad -\delta < x-a < \delta \quad \Rightarrow a-\delta < x < a+\delta$$

$$\sqrt{a-\delta} < \sqrt{x} < \sqrt{a+\delta} \Rightarrow \sqrt{a-\delta}+\sqrt{a} < \sqrt{x}+\sqrt{a} < \sqrt{a+\delta}+\sqrt{a}$$

$$\Rightarrow \sqrt{x}+\sqrt{a} > \sqrt{a-\delta}+\sqrt{a}$$

Let $\delta < a \Rightarrow \quad a-\delta > 0 \quad \Rightarrow \sqrt{x}+\sqrt{a} > \underbrace{\sqrt{a-\delta}}_{>0}+\sqrt{a} > \sqrt{a}$

$$\Rightarrow \frac{1}{\sqrt{x}+\sqrt{a}} < \frac{1}{\sqrt{a}}$$

$$< \quad \frac{|x-a|}{\sqrt{a}} \quad \Rightarrow \quad \frac{\delta}{\sqrt{a}} = \varepsilon \quad \Rightarrow \boxed{\delta = \sqrt{a}\cdot\varepsilon}$$

$$\Rightarrow \lim_{x\to a} \sqrt{x} = \sqrt{a}$$

Then apply Thm 4.1.9 for compositions.

(c) Notice that

$$\max\{f, g\} = \frac{1}{2}(f+g) + \frac{1}{2}|f-g|$$

Then apply Thm 4.1.8 (a) and part (a) of this problem.

(d) Similar to (c)

$$\min\{f, g\} = \frac{1}{2}(f+g) - \frac{1}{2}|f-g|$$

(e) Note that $\lim_{x\to a} x^n = a^n$ (see example 4.1.4 or class notes). Then $[f(x)]^n$ is continuous by Thm 4.1.9

Section 4.3

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$x_n = g(x_{n-1})$ initial value x_3

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} g(x_{n-1}) = g(\lim_{n\to\infty} x_{n-1})$$

$$\Rightarrow p = g(p)$$

Solution

If g has a fixed point and g is continuous, not all sequences will converge.

Ex $f(x) = x^2 - 3$ root $\alpha = \sqrt{3}$

$$g(x) = x - \tfrac{1}{2}(x^2-3) \qquad g'(x) = 1 - x$$

$$|g'(\sqrt{3})| = |1-\sqrt{3}| \approx 0.7 < 1$$

Then sequence $x_n = x_{n-1} - \frac{1}{2}(x_{n-1}^2 - 3)$ will converge with any initial guess.

If we take $g(x) = \frac{3}{x}$ Then $g'(x) = -\frac{3}{x^2}$ and $|g'(\sqrt{3})| = 1$. Starting with $x_3 = 2$, will give us sequence $\{2, \frac{3}{2}, 2, \frac{3}{2}, \dots\}$ which doesn't converge