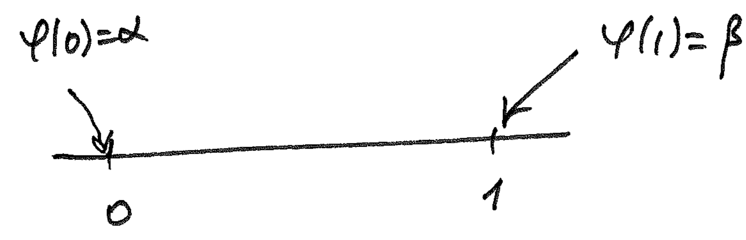
Lecture 12

University of Idaho

2-point boundary value problem

Find $\varphi(x)$, $0 < x < 1$, such that

$$-\varphi'' + c(x)\varphi = f(x)$$
$$\varphi(0) = \alpha$$
$$\varphi(1) = \beta$$



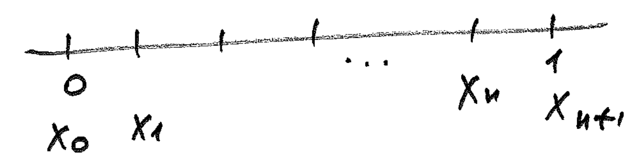
Finite difference scheme

Choose n: integer, $h = \frac{1}{n+1}$: mesh size

$1-0$

$x_i = ih$, $i = 0, \dots, n+1$

$x_0 = 0$, $x_{n+1} = 1$



$\varphi(x_i) = \varphi_i$: exact value

u_i: approximation

University of Idaho

Recall

$$\varphi''(x_i) = \frac{\varphi(x_{i+1}) - 2\varphi(x_i) + \varphi(x_{i-1})}{h^2} + O(h^2)$$

$$\varphi''(x_i) = \frac{\varphi_{i+1} - 2\varphi_i + \varphi_{i-1}}{h^2} + O(h^2)$$

$$-\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + c_i u_i = f_i \qquad \text{where} \quad \begin{aligned} c_i &= c(x_i) \\ f_i &= f(x_i) \end{aligned}$$

$u_1, u_2, \dots, u_n$: unknowns

$u_0 = \alpha, \quad u_{n+1} = \beta$

$i=1$: $$-\frac{u_2 - 2u_1 + \overset{\alpha}{u_0}}{h^2} + c_1 u_1 = f_1$$

$i=n$: $$-\frac{\overset{\beta}{u_{n+1}} - 2u_n + u_{n-1}}{h^2} + c_n u_n = f_n$$

$$\underbrace{\frac{1}{h^2}\begin{pmatrix} 2+c_1h^2 & -1 & & & 0 \\ -1 & 2+c_2h^2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ 0 & & -1 & 2+c_{n-1}h^2 & -1 \\ & & & -1 & 2+c_nh^2 \end{pmatrix}}_{A_h} \underbrace{\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n-1} \\ u_n \end{pmatrix}}_{u} = \underbrace{\begin{pmatrix} f_1+\frac{\alpha}{h^2} \\ f_2 \\ \vdots \\ f_{n-1} \\ f_n+\frac{\beta}{h^2} \end{pmatrix}}_{f}$$

$A_h u = f$: linear system of equations

A_h: tridiagonal, symmetric $\left(\diagdown\!\!\diagdown\!\!\diagdown\right)$

Questions

1. How to solve $A_h u = f$ efficiently?

University of Idaho

2. Is matrix A_h invertible for all h, $c(x)$?

3. Does the method converge? ie.

$$\lim_{h\to 0} \max_{1\le i\le n} |\varphi_i - u_i| = 0 \; ?$$

if so, what is the rate of convergence?

$$\max_{1\le i\le n} |\varphi_i - u_i| = O(h^p) \qquad p - ?$$

<u>LU factorization of a tridiagonal matrix</u>

$$\begin{pmatrix} b_1 & c_1 & & & 0 \\ a_2 & b_2 & c_2 & & \\ & \ddots & \ddots & \ddots & \\ 0 & & a_{n-1} & b_{n-1} & c_{n-1} \\ & & & a_n & b_n \end{pmatrix} =$$

$$= \underbrace{\begin{pmatrix} 1 & & & & 0 \\ l_2 & 1 & & & \\ & l_3 & 1 & & \\ & & \ddots & \ddots & \\ 0 & & & l_n & 1 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} v_1 & c_1 & & & 0 \\ & v_2 & c_2 & & \\ & & \ddots & \ddots & \\ 0 & & & v_{n-1} & c_{n-1} \\ & & & & v_n \end{pmatrix}}_{U}$$

To determine L, U

$b_1 = v_1 \quad \Rightarrow \quad v_1 = b_1$

$a_k = l_k \cdot v_{k-1} \quad \Rightarrow \quad l_k = a_k / v_{k-1}, \quad k = 2, .., n$

$b_k = l_k \cdot c_{k-1} + v_k \quad \Rightarrow \quad v_k = b_k - l_k \cdot c_{k-1}, \quad k = 2, .., n$

To solve $Ly = f$

$$\begin{pmatrix} 1 & & & & 0 \\ l_2 & 1 & & & \\ & l_3 & 1 & & \\ & & \ddots & \ddots & \\ 0 & & & l_n & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_n \end{pmatrix}$$

$y_1 = f_1 \quad \Rightarrow \quad y_1 = f_1$

$l_k y_{k-1} + y_k = f_k \quad \Rightarrow \quad y_k = f_k - l_k y_{k-1}, \quad k = 2, .., n$

To solve $Uu = y$

$$\begin{pmatrix} v_1 & c_1 & & & 0 \\ & v_2 & c_2 & & \\ & & \ddots & \ddots & \\ & & & v_{n-1} & c_{n-1} \\ 0 & & & & v_n \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n-1} \\ u_n \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \\ y_1 \end{pmatrix}$$

$v_n u_n = y_n \quad \Rightarrow \quad u_n = y_n / v_n$

$v_k u_k + c_k u_{k+1} = y_k \quad \Rightarrow \quad u_k = (y_k - c_k u_{k+1}) / v_k$

$k = n-1, \dots, 1$

<u>Operation count</u>

$\# \text{mult} \sim 3n \ll \frac{n^3}{3}$ (HW)

Recall

A is positive definite if $x^T A x > 0$ for $x \neq 0$.

Ex $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ is positive definite

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$x^T A x = (x_1 \quad x_2) \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 2x_1^2 + 2x_2^2 - x_1 x_2 - x_2 x_1 =$$

$$= 2x_1^2 + 2x_2^2 - 2x_1 x_2 = x_1^2 + x_2^2 + \left(x_1^2 - 2x_1 x_2 + x_2^2\right) =$$

$$= x_1^2 + x_2^2 + (x_1 - x_2)^2 \geq 0$$

If $x \neq 0 \Rightarrow x^T A x > 0$

$x^T A x = 0 \Rightarrow x_1^2 + x_2^2 + (x_1 - x_2)^2 = 0 \Rightarrow x_1 = 0,\ x_2 = 0,\ x_1 - x_2 = 0$

$\Rightarrow x = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow$ A is positive definite $\blacksquare$

Ex $A = \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix}$ is NOT positive definite

Proof $x^T A x = x_1^2 + x_2^2 - 4x_1 x_2$

$x = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow x^T A x = 1^2 + 0^2 - 4 \cdot \cancel{1 \cdot 0}^{0} = 1 > 0$

$x = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow x^T A x = 1^2 + 1^2 - 4 \cdot 1 \cdot 1 = -2 < 0$

$\Rightarrow$ A is not positive definite ∎

Claim
If A is a positive definite matrix, then A is invertible.

Pf Recall: A is invertible if $Ax = 0$ has a unique solution $x = 0$.
Consider $Ax = 0$. $x^T A x = 0$, but A is positive definite $\Rightarrow$
$\Rightarrow x = 0 \Rightarrow$ A is invertible. ∎

University of Idaho

Claim

If $c(x) \geq 0$, then matrix A_h arizing in the finite difference approximation scheme is positive definite (and invertible) for all $h > 0$.