Lecture 15

Iterative Methods

$$Ax = b \iff x = Bx + C$$

$$x_{k+1} = \underbrace{B}_{\text{iteration matrix}} x_k + C$$

Jacobi's method (simultaneous displacements)

$A = L + D + U$: matrix splitting

$D = \mathrm{diag}(a_{11}, a_{22}, \dots, a_{nn})$, $\quad a_{ii} \neq 0$, $\quad i = 1, \dots, n$

$$L = \begin{pmatrix} 0 & & & \\ a_{21} & 0 & & 0 \\ \vdots & \ddots & \ddots & \\ a_{n1} & \dots & a_{n,n-1} & 0 \end{pmatrix} \qquad U = \begin{pmatrix} 0 & a_{12} & \dots & a_{1n} \\ & 0 & \dots & \vdots \\ & & \ddots & a_{n-1,n} \\ 0 & & & 0 \end{pmatrix}$$

University of Idaho

$$Ax = b \iff (L + D + U)x = b \qquad\qquad x = Bx + c$$

$$Dx = -(L+U)x + b$$

$$x = \underbrace{-D^{-1}(L+U)}_{B_J}x + D^{-1}b$$

$B_J = -D^{-1}(L+U)$: iteration matrix for Jacobi's method

$$D = \begin{pmatrix} a_{11} & & & 0 \\ & a_{22} & & \\ & & \ddots & \\ 0 & & & a_{nn} \end{pmatrix} \Rightarrow D^{-1} = \begin{pmatrix} \frac{1}{a_{11}} & & & 0 \\ & \frac{1}{a_{22}} & & \\ & & \ddots & \\ 0 & & & \frac{1}{a_{nn}} \end{pmatrix} = \mathrm{diag}\left(\frac{1}{a_{11}}, \dots, \frac{1}{a_{nn}}\right)$$

In practice we will use

$$Dx_{k+1} = -(L+U)x_k + b$$: easy to solve for x_{k+1}

Components

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

$$a_{11} x_1^{(k+1)} = -a_{12} x_2^{(k)} - a_{13} x_3^{(k)} + b_1$$

$$a_{22} x_2^{(k+1)} = -a_{21} x_1^{(k)} - a_{23} x_3^{(k)} + b_2$$

$$a_{33} x_3^{(k+1)} = -a_{31} x_1^{(k)} - a_{32} x_2^{(k)} + b_3$$

In general,

$$x_i^{(k+1)} = \frac{1}{a_{ii}} \left[b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k)} \right]$$

Ex

$$2x_1 - x_2 = 1$$

$$-x_1 + 2x_2 = 1$$

$x_1 = 1$, $x_2 = 1$: exact solution $\quad x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$2x_1^{(k+1)} = x_2^{(k)} + 1$$

$$2x_2^{(k+1)} = x_1^{(k)} + 1$$

$$\Rightarrow \quad x_1^{(k+1)} = \frac{1}{2}\left(x_2^{(k)} + 1\right)$$

$$x_2^{(k+1)} = \frac{1}{2}\left(x_1^{(k)} + 1\right)$$

k	$x_1^{(k)}$	$x_2^{(k)}$
0	0	0
1	$\frac{1}{2}$	$\frac{1}{2}$
2	$\frac{3}{4}$	$\frac{3}{4}$
3	$\frac{7}{8}$	$\frac{7}{8}$

arbitrary guess

converges to $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Define $e_k = x - x_k$: error

Then
$$\|e_0\|_\infty = 1$$
$$\|e_1\|_\infty = \frac{1}{2}$$
$$\|e_2\|_\infty = \frac{1}{4}$$
$$\|e_3\|_\infty = \frac{1}{8}$$

$\Rightarrow \|e_{k+1}\|_\infty = \frac{1}{2}\|e_k\|$ for $k \geq 0$

$Ax = b$

$x = Bx + c$

$x_{k+1} = Bx_k + c$

$e_k = x - x_k$: error

<u>Thm</u> If $\|B\| < 1$ for some subordinate matrix norm, then $x_k \to x$ for any initial guess x_0.

<u>Proof</u>

$e_k = x - x_k = Bx + c - (Bx_{k-1} + c) = B(x - x_{k-1}) = Be_{k-1}$

$\Rightarrow e_k = Be_{k-1} = B^2 e_{k-2} = \dots = B^k e_0$

$e_k = B^k e_0$

$\|e_k\| = \|B^k e_0\| \leq \|B^k\| \cdot \|e_0\|$

$\|B^2\| = \|B \cdot B\| \leq \|B\| \cdot \|B\| = \|B\|^2 \Rightarrow \|B^k\| \leq \|B\|^k$

$\therefore \|e_k\| \leq \|B^k\| \cdot \|e_0\| \leq \|B\|^k \cdot \|e_0\| \to 0$ as $k \to \infty$ since $\|B\| < 1$

$\Rightarrow \|e_k\| \to 0$ as $k \to \infty \Rightarrow \|x - x_k\| \to 0$ as $k \to \infty \Rightarrow x - x_k \to 0$
$\Rightarrow x_k \to x$ as $k \to \infty$ ∎

Important: $e_{k+1} = B e_k \Rightarrow \|e_{k+1}\| \leq \|B\| \cdot \|e_k\|$

Ex $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ $\quad B_J = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$

Note $\|B_J\|_\infty = \frac{1}{2} < 1$

$\Rightarrow$ we know that Jacobi's method converges.

Gauss-Seidel Method (successive displacements)

$A = L + D + U$: as before

$$Ax = b \iff (L + D + U)x = b$$

$$(L + D)x = -Ux + b$$

$$x = \underbrace{-(L+D)^{-1}U}_{B_{GS}} x + (L+D)^{-1} b$$

B_{GS} : iteration matrix for Gauss-Seidel method

In practice:

$$(L+D)\,x_{k+1} = -\bar{U}\,x_k + b \;:\; \text{easy to solve for } x_{k+1}$$

<u>Components</u>

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

$$a_{11}x_1^{(k+1)} = -a_{12}x_2^{(k)} - a_{13}x_3^{(k)} + b_1$$

$$a_{22}x_2^{(k+1)} = -a_{21}x_1^{(k+1)} - a_{23}x_3^{(k)} + b_2$$

$$a_{33}x_3^{(k+1)} = -a_{31}x_1^{(k+1)} - a_{32}x_2^{(k+1)} + b_3$$

<u>Note</u>. To compute $x_i^{(k+1)}$, we use already updated components $x_1^{(k+1)}, \ldots, x_{i-1}^{(k+1)}$, i.e. we use $x_i^{(k+1)}$ as soon as it becomes available.

In general,

$$x_i^{(k+1)} = \frac{1}{a_{ii}} \left[b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k)} \right]$$

Ex

$$2x_1 - x_2 = 1$$

$$-x_1 + 2x_2 = 1$$

$$2x_1^{(k+1)} = x_2^{(k)} + 1$$

$$2x_2^{(k+1)} = x_1^{(k+1)} + 1$$

$\Rightarrow$

$$x_1^{(k+1)} = \frac{1}{2}\left[x_2^{(k)} + 1\right]$$

$$x_2^{(k+1)} = \frac{1}{2}\left[x_1^{(k+1)} + 1\right]$$

k	$x_1^{(k)}$	$x_2^{(k)}$
0	0	0
1	$\frac{1}{2}$	$\frac{3}{4}$
2	$\frac{7}{8}$	$\frac{15}{16}$
3	$31/32$	$63/64$

– converges faster than Jacobi

$\|e_0\|_\infty = 1$

$\|e_1\|_\infty = \frac{1}{2}$

$\|e_2\|_\infty = \frac{1}{8}$

$\|e_3\|_\infty = \frac{1}{32}$

In general, $\|e_{k+1}\|_\infty = \frac{1}{4}\|e_k\|_\infty$ for $k \geq 1$

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \qquad B_{GS} = -(L+D)^{-1}U = \begin{pmatrix} 0 & \frac{1}{2} \\ 0 & \frac{1}{4} \end{pmatrix}$$

$\|B_{GS}\| = \frac{1}{2} < 1 \Rightarrow$ G.-S. converges

<u>Summary</u>

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

$$B_J = -D^{-1}(L+U) = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \Rightarrow \|B_J\|_\infty = \frac{1}{2}$$

$$B_{GS} = -(L+D)^{-1}\bar{U} = \begin{pmatrix} 0 & \frac{1}{2} \\ 0 & \frac{1}{4} \end{pmatrix} \Rightarrow \|B_{GS}\|_\infty = \frac{1}{2}$$

In both cases, we proved that $\|e_{k+1}\|_\infty \leq \frac{1}{2}\|e_k\|_\infty$.

In fact, we showed that

$$\|e_{k+1}\|_\infty = \frac{1}{2}\|e_k\|_\infty \quad \text{for Jacobi}$$

$$\|e_{k+1}\|_\infty = \frac{1}{4}\|e_k\|_\infty \quad \text{for Gauss-Seidel}$$