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Lecture 17

Relaxation

$Ax = b \qquad A = L + D + U$

$$Ax = b \iff (L + D + U)x = b$$

$$(L + D)x = -Ux + b$$

Gauss-Seidel method:

$$(L + D)x_{k+1} = -Ux_k + b$$

One of the forms of Gauss-Seidel method is

$$Dx_{k+1} = \underline{Dx_k} - (Lx_{k+1} + (\underline{D} + U)x_k - b)$$

Let ω be an acceleration parameter

$$\boxed{Dx_{k+1} = Dx_k - \omega\left(Lx_{k+1} + (D + U)x_k - b\right)} \leftarrow \text{used in practice}$$

$$(\omega L + D)x_{k+1} = Dx_k - \omega(D + U)x_k + \omega b$$

$$(\omega L + D)x_{k+1} = \left((1 - \omega)D - \omega U\right)x_k + \omega b$$

$$B_\omega = (\omega L + D)^{-1}\left((1-\omega)D - \omega \bar{U}\right) \text{ : iteration matrix}$$

Note when $\omega = 1$ we recover Gauss-Seidel

Components

$$\begin{array}{ccc} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{array} \leftarrow U$$

D L

$$a_{11} x_1^{(k+1)} = a_{11} x_1^{(k)} - \omega\left(a_{11} x_1^{(k)} + a_{12} x_2^{(k)} + a_{13} x_3^{(k)} - b_1\right)$$

$$a_{22} x_2^{(k+1)} = a_{22} x_2^{(k)} - \omega\left(a_{21} x_1^{(k+1)} + a_{22} x_2^{(k)} + a_{23} x_3^{(k)} - b_2\right)$$

$$a_{33} x_3^{(k+1)} = a_{33} x_3^{(k)} - \omega\left(a_{31} x_1^{(k+1)} + a_{32} x_2^{(k+1)} + a_{33} x_3^{(k)} - b_3\right)$$

Note
When $1 < \omega < 2$, the method is called successive overrelaxation (SOR). It is used to accelerate converge for those systems that converge using Gauss-Seidel method.

$0 < \omega < 1$: <u>under-relaxation methods</u> used to obtain convergence for those systems for which Gauss-Seidel method does not converge.

Ex

$$2x_1 - x_2 = 1$$
$$-x_1 + 2x_2 = 1$$

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 1 \end{pmatrix} : \text{exact sol}^{n}$$

$$2x_1^{(k+1)} = 2x_1^{(k)} - \omega\left(2x_1^{(k)} - x_2^{(k)} - 1\right)$$

$$2x_2^{(k+1)} = 2x_2^{(k)} - \omega\left(-x_1^{(k+1)} + 2x_2^{(k)} - 1\right)$$

$$\begin{pmatrix} 2 & 0 \\ -\omega & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_{k+1} = \begin{pmatrix} 2(1-\omega) & \omega \\ 0 & 2(1-\omega) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_k + \omega\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$B_\omega = \begin{pmatrix} 2 & 0 \\ -\omega & 2 \end{pmatrix}^{-1}\begin{pmatrix} 2(1-\omega) & \omega \\ 0 & 2(1-\omega) \end{pmatrix} = \begin{pmatrix} 1-\omega & \frac{\omega}{2} \\ \frac{\omega(1-\omega)}{2} & \frac{\omega^2}{4} + 1 - \omega \end{pmatrix}$$

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<u>Note</u>: $\omega = 1 \Rightarrow B = \begin{pmatrix} 0 & \frac{1}{2} \\ 0 & \frac{1}{4} \end{pmatrix} = B_{GS} \Rightarrow \rho(B_{GS}) = \frac{1}{4} = \underline{\underline{0.25}}$ (Gauss-Seidel)

$$\omega = \frac{4}{2+\sqrt{3}} \sim 1.0718 \Rightarrow B_\omega = \begin{pmatrix} -0.0718 & 0.5359 \\ -0.0385 & 0.2154 \end{pmatrix}$$

$$\Rightarrow \rho(B_\omega) = \underline{\underline{0.07}}$$

$\|e_k\|_\infty$ for G.-S.	k	$x_1^{(k)}$	$x_2^{(k)}$	$\|e_k\|_\infty$	$\|e_{k+1}\|_\infty / \|e_k\|_\infty$
1.0000	0	0.0000	0.0000	1.0000	0.4641
$\frac{1}{2} = 0.5$	1	0.5359	0.8231	0.4641	0.1325
$\frac{1}{8} = 0.125$	2	0.9385	0.9798	0.0615	0.1047
$\frac{1}{32} = 0.03125$	3	0.9936	0.9980	0.0064	0.0944
$\frac{1}{128} = 0.00731$	4	0.9994	0.9998	0.0006	$\downarrow$ 0.07

Thm

If $\rho(B_\omega) < 1$, then $0 < \omega < 2$.

Pf

We will prove this result for our example of 2×2 matrix.

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \qquad B_\omega = \begin{pmatrix} 1-\omega & \frac{\omega}{2} \\ \frac{\omega(1-\omega)}{2} & \frac{\omega^2}{4} + 1 - \omega \end{pmatrix}$$

We will need to use a result about matrices:

$\det A = \lambda_1 \lambda_2 \dots \lambda_n$, where $\lambda_1, \lambda_2, \dots, \lambda_n$ are e'values of A.

Let λ_1, λ_2 be e'values of B_ω.

If $\rho(B_\omega) < 1 \Rightarrow |\lambda_1| < 1, \ |\lambda_2| < 1 \Rightarrow |\underbrace{\lambda_1 \lambda_2}_{\det B_\omega}| < 1$

$\Rightarrow |\det B_\omega| < 1$

$$\det B_\omega = (1-\omega)\left(\frac{\omega^2}{4} + 1 - \omega\right) - \frac{\omega}{2} \cdot \frac{\omega(1-\omega)}{2} = (1-\omega)\left(\cancel{\frac{\omega^2}{4}} + 1 - \omega - \cancel{\frac{\omega^2}{4}}\right)$$
$$= (1-\omega)^2$$

$|\det B_\omega| < 1 \Rightarrow (1-\omega)^2 < 1 \Rightarrow |1-\omega| < 1$ or $|\omega - 1| < 1$

$-1 < \omega - 1 < 1 \quad | +1$

$\boxed{0 < \omega < 2}$

∎

Thm

Let A be block tridiagonal, symmetric, positive definite.

Define

$$\omega_* = \frac{2}{1+\sqrt{1-\rho(B_J)^2}} : \text{optimal SOR parameter}$$

Then

$$\rho(B_{\omega*}) = \min_{0<\omega<2} \rho(B_\omega) = \omega_* - 1 < \rho(B_{GS}) < \rho(B_J) < 1$$

Ex $\rho(B_J) = \frac{1}{2} \Rightarrow \omega_* = \dfrac{2}{1+\sqrt{1-\left(\frac{1}{2}\right)^2}} = \dfrac{4}{2+\sqrt{3}} \approx 1.0718$