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Lecture 21

$$f(x) = \frac{1}{1+25x^2} = \frac{1}{1-(-25x^2)}$$

$$P_{2n}(x) = \sum_{k=0}^{n} (-25x^2)^k = 1 - 25x^2 + 625x^4 - 15625x^6 + \dots + (-25x^2)^n$$

$$\frac{1}{1-z} = 1 + z + z^2 + \dots$$: geometric series converges for $|z| < 1$

$$z = -25x^2 \Rightarrow |-25x^2| < 1 \Rightarrow x^2 < \frac{1}{25} \Rightarrow |x| < \frac{1}{5} = 0.2$$

Note

1. if $|x| < 0.2 \Rightarrow \lim_{n\to\infty} P_{2n} = f(x)$
2. if $|x| \not< 0.2 \Rightarrow \lim_{n\to\infty} P_{2n}$ does not exist

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Polynomial interpolation

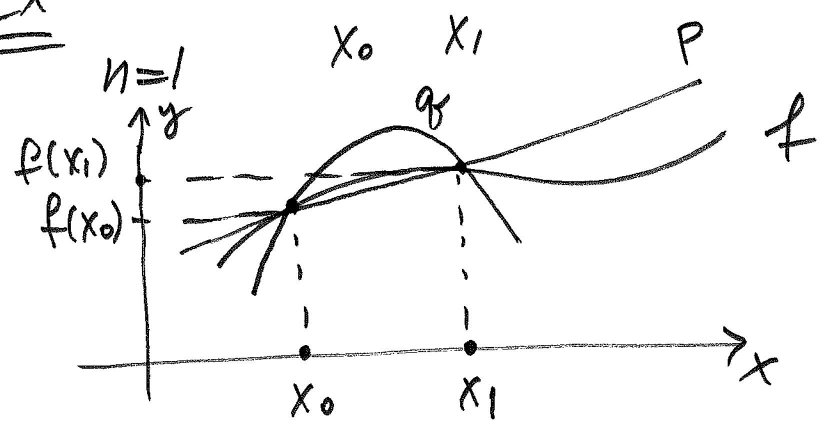
Let f be a continuous function and $x_0, x_1, \ldots, x_n$: distinct points.

Questions

1. Does there exist a unique polynomial p of least degree which interpolates function at given points, i.e. such that

$f(x_i) = p(x_i), \quad i = 0, 1, \ldots, n$?

Ex



$\deg p = 1$

$\deg q = 2$

Thm (uniqueness)

If $x_0, x_1, \ldots, x_n$ are $n+1$ distinct points and p, q are polynomials of degree $\le n$ such that $p(x_i) = q(x_i)$, for $i = 0, \ldots, n$, then $p(x) = q(x)$ for all x.

Pf

Use fundamental theorem of algebra (n^{th} degree polynomial has exactly n roots). Here we consider

$h(x) = p(x) - q(x) \Rightarrow$

$h(x_i) = 0, \quad i = 0, 1, \ldots, n : n+1$ points

2. What is the best way to evaluate $p(x)$ at $x \ne x_i$?
3. How large is the error $|f(x) - p(x)|$ at $x \ne x_i$?

Def Let $\mathcal{P}_n$ the set of polynomials of degree $\le n$.

$$\mathcal{P}_n = \{ a_0 + a_1 x + \ldots + a_k x^k, \quad k \le n, \quad a_i \in \mathbb{R} \}$$

Note $\mathcal{P}_n$ is a vector space over $\mathbb{R}$ since if $p, q \in \mathcal{P}_n$

$\Rightarrow p+q \in \mathcal{P}_n$ and $\alpha p \in \mathcal{P}_n$, $\alpha \in \mathbb{R}$

$\dim \mathcal{P}_n = n+1$

The standard basis for $\mathcal{P}_n$ is $\{1, x, x^2, \dots, x^n\}$

$p(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$: in standard basis

Def $$\ell_k(x) = \prod_{\substack{i=0 \\ i \neq k}}^{n} \frac{x - x_i}{x_k - x_i}, \quad k = 0, 1, \dots, n : \quad \underline{\text{Lagrange polynomials}}$$

Ex $n=2$ $\quad x_0 = 1, \quad x_1 = 2, \quad x_2 = 3$

$$\ell_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{(x-2)(x-3)}{(1-2)(1-3)} = \frac{1}{2}x^2 - \frac{5}{2}x + 3$$

$$\ell_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{(x-1)(x-3)}{(2-1)(2-3)} = -x^2 + 4x - 3$$

$$l_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{(x-1)(x-2)}{(3-1)(3-2)} = \frac{1}{2}x^2 - \frac{3}{2}x + 1$$

<u>Note</u>

1. $\deg l_k(x) = n$
2. $l_k(x_i) = \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases}$

Given f: continuous, $x_0, x_1, \ldots, x_n$: $n+1$ distinct points, define

$$\boxed{p_n(x) = \sum_{k=0}^{n} f(x_k) \cdot l_k(x) = f(x_0)\, l_0(x) + f(x_1)\, l_1(x) + \ldots + f(x_n)\, l_n(x)}$$

<u>interpolating polynomial in Lagrange form</u>

<u>Note</u>

1. $\deg p_n \leq n$
2. $p_n(x_i) = \sum_{k=0}^{n} f(x_k) \cdot \underbrace{l_k(x_i)}_{= \begin{smallmatrix} 1 \\ 0 \end{smallmatrix} \text{ if } \begin{smallmatrix} i = k \\ i \neq k \end{smallmatrix}} = f(x_i), \quad i = 0, 1, \ldots, n$

Thus, $p_n(x)$ interpolates $f(x)$ at $x_0, x_1, \dots, x_n$ and has degree $\leq n$.

<u>Ex</u> $f(x) = \frac{1}{x}$, $x_0 = 1$, $x_1 = 2$, $x_2 = 3$, $n = 2$

$$P_2(x) = f(x_0)\,l_0(x) + f(x_1)\,l_1(x) + f(x_2)\,l_2(x) =$$

$$= 1 \cdot \left(\frac{1}{2}x^2 - \frac{5}{2}x + 3\right) + \frac{1}{2}\left(-x^2 + 4x - 3\right) + \frac{1}{3}\left(\frac{1}{2}x^2 - \frac{3}{2}x + 1\right) =$$

$$= \frac{1}{6}x^2 - x + \frac{11}{6}$$

$$P_2(1) = \frac{1}{6} - 1 + \frac{11}{6} = 1 \quad \checkmark$$

$$P_2(2) = \frac{1}{6} \cdot 2^2 - 2 + \frac{11}{6} = \frac{4}{6} - 2 + \frac{11}{6} = \frac{1}{2} \quad \checkmark$$

$$P_2(3) = \frac{1}{3} \quad \checkmark$$

Note

Lagrange form of interpolating polynomial $p_n(x)$ is good because it shows that interpolating polynomial exists, but it has computational disadvantages:

1. it is expensive to evaluate $p_n(x)$ at $x \neq x_i$
2. if we want to include an additional point x_{n+1}, we have to recompute all Lagrange polynomials $l_k(x)$.

Def

$$\boxed{p_n(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + \ldots + a_n(x-x_0)(x-x_1)\ldots(x-x_{n-1})}$$

This is an <u>interpolating polynomial in Newton's form</u>

<u>Ex</u> $n=1$ x_0, x_1

$$p_1(x) = f(x_0)\,l_0(x) + f(x_1)\,l_1(x) = f(x_0)\frac{x-x_1}{x_0-x_1} + f(x_1)\frac{x-x_0}{x_1-x_0}:$$

<u>Lagrange form</u>

$$= \underbrace{f(x_0)}_{a_0} + \underbrace{\frac{f(x_1)-f(x_0)}{x_1-x_0}}_{a_1}\cdot(x-x_0):$$ <u>Newton's form</u>