

Note If $n > 1$, we need to have an algorithm of computing coefficients a_0, a_1, \dots, a_n in Newton's form of interpolating polynomial P_n .

Claim There exists a number a_n such that

$$P_n(x) = P_{n-1}(x) + a_n(x-x_0)(x-x_1)\dots(x-x_{n-1})$$

Pf

$$P_n(x_i) = P_{n-1}(x_i) + a_n(x_i-x_0)(x_i-x_1)\dots(x_i-x_{n-1})$$

$$\text{If } 0 \leq i \leq n-1 \Rightarrow P_n(x_i) = P_{n-1}(x_i) + a_n \cdot 0$$

For $i=n$:

$$P_n(x_n) = P_{n-1}(x_n) + a_n \underbrace{(x_n-x_0)(x_n-x_1)\dots(x_n-x_{n-1})}_{\neq 0} : \begin{array}{l} \text{gives} \\ \text{equation} \\ \text{to compute} \\ a_n \end{array}$$

$$\therefore a_n = \frac{f(x_n) - p_{n-1}(x_n)}{(x_n - x_0)(x_n - x_1) \dots (x_n - x_{n-1})}$$

Notation

$$p_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \dots + a_n(x - x_0)(x - x_1) \dots (x - x_{n-1})$$

$$a_0 = f[x_0]$$

$$a_1 = f[x_0, x_1]$$

$$a_2 = f[x_0, x_1, x_2]$$

...

$$a_n = f[x_0, x_1, \dots, x_n]$$

Then

$$p_n(x) = f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) + \dots \\ \dots + f[x_0, x_1, \dots, x_n](x - x_0)(x - x_1) \dots (x - x_{n-1})$$

Newton's form of interpolating polynomial

Note

It is easy to include an additional point x_{n+1} . The previous work will not be wasted, i.e. previous terms will not be changed.

Claim $f[x_0, x_1, \dots, x_n] = \frac{f[x_1, \dots, x_n] - f[x_0, \dots, x_{n-1}]}{x_n - x_0}$: divided differences

Proof

$p_{n-1}(x)$: interpolates f at x_0, x_1, \dots, x_{n-1} : $\deg p_{n-1} \leq n-1$

$g_{n-1}(x)$: interpolates f at x_1, x_2, \dots, x_n : $\deg g_{n-1} \leq n-1$

Define

$$s(x) = \frac{x - x_0}{x_n - x_0} g_{n-1}(x) + \frac{x_n - x}{x_n - x_0} p_{n-1}(x)$$

$$\deg s \leq n$$

$$S(x_0) = p_{n-1}(x_0) = f(x_0)$$

$$S(x_n) = p_{n-1}(x_n) = f(x_n)$$

For $1 \leq i \leq n-1$

$$S(x_i) = \frac{x_i - x_0}{x_n - x_0} \underbrace{p_{n-1}(x_i)}_{f(x_i)} + \frac{x_n - x_i}{x_n - x_0} \underbrace{p_{n-1}(x_i)}_{f(x_i)} = f(x_i) \left(\frac{x_i - x_0}{x_n - x_0} + \frac{x_n - x_i}{x_n - x_0} \right) = 1$$

$$= f(x_i) \quad \checkmark$$

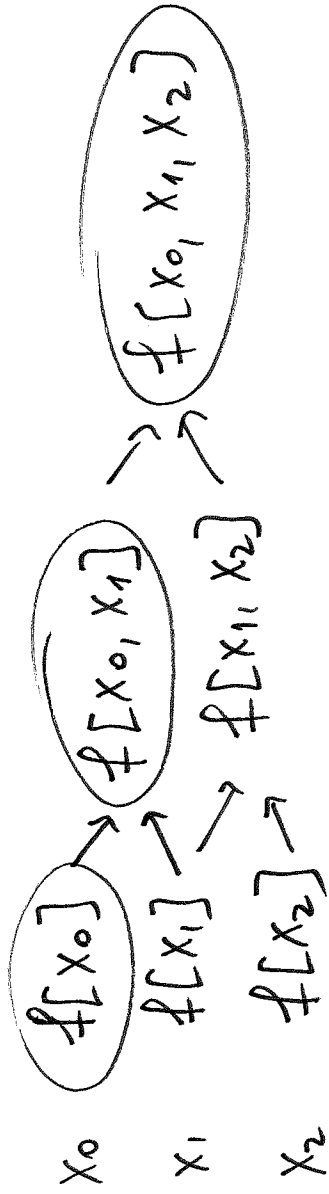
$\Rightarrow S(x)$ interpolates f at x_0, x_1, \dots, x_n

$\Rightarrow S(x) = p_n(x)$ by uniqueness

Evaluate coefficients of x^n .

$$\frac{1}{x_n - x_0} f[x_1, x_2, \dots, x_n] - \frac{1}{x_n - x_0} f[x_0, x_1, \dots, x_{n-1}] = f[x_0, x_1, \dots, x_n]$$

$$\Rightarrow f[x_0, x_1, \dots, x_n] = \frac{f[x_1, x_2, \dots, x_n] - f[x_0, x_1, \dots, x_{n-1}]}{x_n - x_0}$$

Divided difference table

$$f[x_0] = f(x_0), \quad f[x_1] = f(x_1), \quad f[x_2] = f(x_2)$$

$$f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0}$$

$$f[x_1, x_2] = \frac{f[x_2] - f[x_1]}{x_2 - x_1}$$

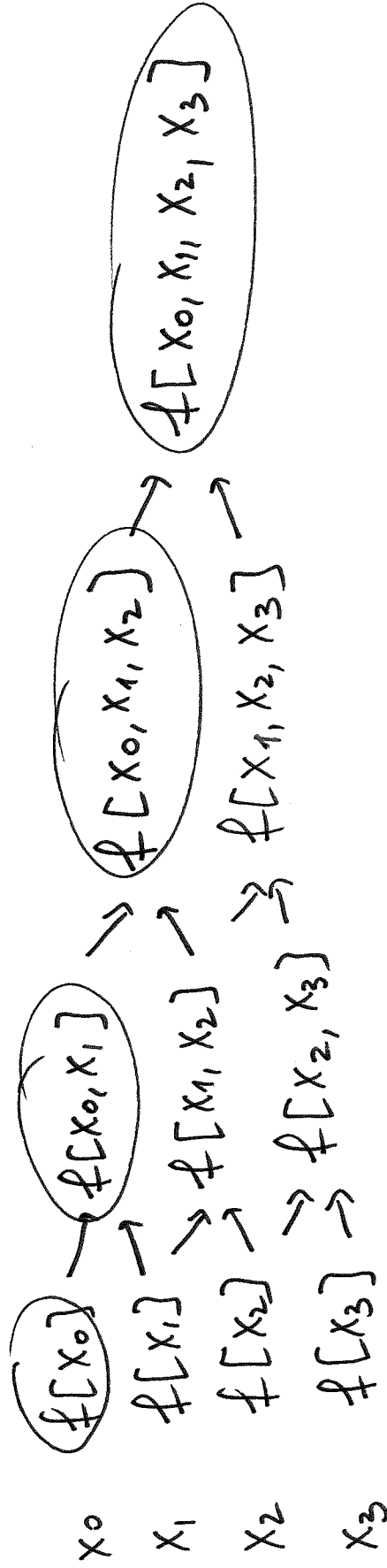
$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$

Then

$$p_2(x) = f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1)$$

i.e. the circled numbers are the coefficients of the interpolating polynomial $p_2(x)$ in Newton's form.

If we want to add an extra point x_3 , then



$$p_3(x) = f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) + f[x_0, x_1, x_2, x_3](x - x_0)(x - x_1)(x - x_2)$$

Ex $f(x) = \frac{1}{x}$, $x_0 = 1$, $x_1 = 2$, $x_2 = 3$

$$x_i \quad f[x_i] \quad f[x_0, \dots, x_i]$$

$$1 \quad \textcircled{1}$$

$$\frac{\frac{1}{2} - 1}{2 - 1} = \textcircled{-\frac{1}{2}}$$

$$\frac{-\frac{1}{6} - (-\frac{1}{2})}{3 - 1} = \textcircled{\frac{1}{6}}$$

$$2 \quad \frac{1}{2}$$

$$\frac{\frac{1}{3} - \frac{1}{2}}{3 - 2} = -\frac{1}{6}$$

$$3 \quad \frac{1}{3}$$

$$p_2(x) = 1 - \frac{1}{2}(x-1) + \frac{1}{6}(x-1)(x-2)$$

evaluation of $p_n(x)$

Newton's form
3 mult

$$p_2(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) :$$

$$p_2(x) = a_0 + (x-x_0)(a_1 + a_2(x-x_1)) : \quad \text{nested form}$$

2 mult

General case

$$p_n(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + \dots + a_n(x-x_0)(x-x_1)\dots(x-x_{n-1})$$

Newton's form

$$p_n(x) = a_0 + (x-x_0) \left(a_1 + (x-x_1) \left(a_2 + \dots + a_n (x-x_{n-1}) \right) \right)$$

nested form