Lecture 22

Note

If $n > 1$, we need to have an algorithm of computing coefficients $a_0, a_1, \dots, a_n$ in Newton's form of interpolating polynomial p_n.

Claim

There exists a number a_n such that

$$p_n(x) = p_{n-1}(x) + a_n (x - x_0)(x - x_1) \dots (x - x_{n-1})$$

Pf

$$p_n(x_i) = p_{n-1}(x_i) + a_n (x_i - x_0)(x_i - x_1) \dots (x_i - x_{n-1})$$

If $0 \le i \le n-1 \Rightarrow p_n(x_i) = p_{n-1}(x_i) + a_n \cdot 0$

For $i = n$:

$$\underbrace{p_n(x_n)}_{\overset{\shortparallel}{f(x_n)}} = p_{n-1}(x_n) + a_n \underbrace{(x_n - x_0)(x_n - x_1) \dots (x_n - x_{n-1})}_{\neq 0} :$$

gives equation to compute a_n

$$\therefore \quad a_n = \frac{f(x_n) - p_{n-1}(x_n)}{(x_n - x_0)(x_n - x_1)\dots(x_n - x_{n-1})}$$

Notation

$$p_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \dots + a_n(x - x_0)(x - x_1)\dots(x - x_{n-1})$$

$$a_0 = f[x_0]$$
$$a_1 = f[x_0, x_1]$$
$$a_2 = f[x_0, x_1, x_2]$$
$$- \quad - \quad - \quad -$$
$$a_n = f[x_0, x_1, \dots, x_n]$$

Then

$$\boxed{\begin{aligned} p_n(x) = {} & f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) + \dots \\ & \dots + f[x_0, x_1, \dots, x_n](x - x_0)(x - x_1)\dots(x - x_{n-1}) \end{aligned}}$$

Newton's form of interpolating polynomial

Note

It is easy to include an additional point x_{n+1}. The previous work will not be wasted, i.e. previous terms will not be changed.

Claim

$$f[x_0, x_1, \dots, x_n] = \frac{f[x_1, \dots, x_n] - f[x_0, \dots, x_{n-1}]}{x_n - x_0} \quad : \quad \underline{\text{divided differences}}$$

Proof

$p_{n-1}(x)$: interpolates f at $x_0, x_1, \dots, x_{n-1}$: $\deg p_{n-1} \le n-1$

$q_{n-1}(x)$: interpolates f at $x_1, x_2, \dots, x_n$: $\deg q_{n-1} \le n-1$

Define

$$s(x) = \frac{x - x_0}{x_n - x_0} q_{n-1}(x) + \frac{x_n - x}{x_n - x_0} p_{n-1}(x)$$

$\deg s \le n$

$$s(x_0) = p_{n-1}(x_0) = f(x_0)$$

$$s(x_n) = q_{n-1}(x_n) = f(x_n)$$

For $1 \leq i \leq n-1$

$$s(x_i) = \frac{x_i - x_0}{x_n - x_0} \underbrace{q_{n-1}(x_i)}_{f(x_i)} + \frac{x_n - x_i}{x_n - x_0} \underbrace{p_{n-1}(x_i)}_{f(x_i)} = f(x_i) \underbrace{\left(\frac{\cancel{x_i} - x_0}{x_n - x_0} + \frac{x_n - \cancel{x_i}}{x_n - x_0} \right)}_{=1}$$

$$= f(x_i) \quad \checkmark$$

$\Rightarrow$ $s(x)$ interpolates f at $x_0, x_1, \ldots, x_n$

$\Rightarrow$ $s(x) = p_n(x)$ by uniqueness

Evaluate coefficients of x^n.

$$\frac{1}{x_n - x_0} f[x_1, x_2, \ldots, x_n] - \frac{1}{x_n - x_0} f[x_0, x_1, \ldots, x_{n-1}] = f[x_0, x_1, \ldots, x_n]$$

$$\Rightarrow \quad f[x_0, x_1, \ldots, x_n] = \frac{f[x_1, x_2, \ldots, x_n] - f[x_0, x_1, \ldots, x_{n-1}]}{x_n - x_0}$$

$\blacksquare$

Divided difference table

$$
\begin{array}{llll}
x_0 & f[x_0] & & \\
 & & f[x_0, x_1] & \\
x_1 & f[x_1] & & f[x_0, x_1, x_2] \\
 & & f[x_1, x_2] & \\
x_2 & f[x_2] & &
\end{array}
$$

$f[x_0] = f(x_0)$, $\quad f[x_1] = f(x_1)$, $\quad f[x_2] = f(x_2)$

$$f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0}$$

$$f[x_1, x_2] = \frac{f[x_2] - f[x_1]}{x_2 - x_1}$$

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$

Then

$$p_2(x) = f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1)$$

i.e. the circled numbers are the coefficients of the interpolating polynomial $p_2(x)$ in Newton's form.

If we want to add an extra point x_3, then

$$\begin{array}{llllll}
x_0 & f[x_0] & & & & \\
 & & f[x_0,x_1] & & & \\
x_1 & f[x_1] & & f[x_0,x_1,x_2] & & \\
 & & f[x_1,x_2] & & f[x_0,x_1,x_2,x_3] & \\
x_2 & f[x_2] & & f[x_1,x_2,x_3] & & \\
 & & f[x_2,x_3] & & & \\
x_3 & f[x_3] & & & &
\end{array}$$

$$p_3(x) = f[x_0] + f[x_0,x_1](x-x_0) + f[x_0,x_1,x_2](x-x_0)(x-x_1) + f[x_0,x_1,x_2,x_3](x-x_0)(x-x_1)(x-x_2)$$

<u>Ex</u> $f(x)=\frac{1}{x}$, $x_0=1$, $x_1=2$, $x_2=3$

x_i	$f[.]$	$f[.,.]$	$f[.,.,.]$
1	1		
		$\frac{\frac{1}{2}-1}{2-1} = -\frac{1}{2}$	
2	$\frac{1}{2}$		$\frac{-\frac{1}{6}-\left(-\frac{1}{2}\right)}{3-1} = \frac{1}{6}$
		$\frac{\frac{1}{3}-\frac{1}{2}}{3-2} = -\frac{1}{6}$	
3	$\frac{1}{3}$		

$$p_2(x) = 1 - \frac{1}{2}(x-1) + \frac{1}{6}(x-1)(x-2)$$

<u>evaluation of $p_n(x)$</u>

$$p_2(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) :$$ $\frac{\text{Newton's form}}{\text{3 mult}}$

$$p_2(x) = a_0 + (x-x_0)\left(a_1 + a_2(x-x_1)\right) :$$ $\frac{\text{nested form}}{\text{2 mult}}$

General case

$$p_n(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + \dots + a_n(x-x_0)(x-x_1)\dots(x-x_{n-1}):$$

Newton's form

$$p_n(x) = a_0 + (x-x_0)\Big(a_1 + (x-x_1)\big(a_2 + \dots + a_n(x-x_{n-1})\dots\big)\Big):$$

nested form