Lecture 23

Operation count

\# mult $= 1+2+\dots+n = \frac{n(n+1)}{2}$: Newton's form

n : nested form

Code (for nested multiplication)

```
a(1), a(2),..., a(n+1)
x(1), x(2),..., x(n+1)

p = a(n+1)
for i = n : -1 : 1
    p = a(i) + p*(x - x(i))
end
```

Matrix interpretation of $p_n(x_i) = f(x_i), \quad i = 0, 1, \dots, n$

$\mathcal{P}_n$: vector space of polynomials of $\deg \le n$

Standard basis : $\{1, x, x^2, \dots, x^n\}$

$$p_n(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$$

Goal is to find coefficients $a_0, a_1, \dots, a_n$: $n+1$ unknowns.

We need to use $n+1$ conditions, i.e.

$$p_n(x_i) = f(x_i), \quad i = 0, 1, \dots, n$$

$$\underbrace{\begin{pmatrix} 1 & x_0 & x_0^2 & \dots & x_0^n \\ 1 & x_1 & x_1^2 & \dots & x_1^n \\ - & - & - & - & - \\ 1 & x_n & x_n^2 & \dots & x_n^n \end{pmatrix}}_{\text{Vandermonde matrix}} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_n) \end{pmatrix}$$

If $x_0, x_1, \ldots, x_n$ are $n+1$ distinct points, then coefficient matrix (Vandermonde matrix) is invertible. However, it is ill-conditioned, in general.

<u>Lagrange basis</u>: $\{ l_0(x), l_1(x), \ldots, l_n(x) \}$

$$p_n(x) = a_0 l_0(x) + a_1 l_1(x) + \ldots + a_n l_n(x)$$

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ - & - & - & - & - \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} f(x_0) \\ f(x_1) \\ f(x_2) \\ \vdots \\ f(x_n) \end{pmatrix}$$

$\Rightarrow a_0 = f(x_0),\ a_1 = f(x_1),\ \ldots,\ a_n = f(x_n)$

<u>Note</u>: coefficient matrix is identity matrix!

<u>Newton's basis</u>: $\{1,\ x-x_0,\ (x-x_0)(x-x_1), \ldots, (x-x_0)\ldots(x-x_{n-1})\}$

$$p_n(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + \ldots + a_n(x-x_0)\ldots(x-x_{n-1})$$

$$\begin{pmatrix} 1 & & & & 0 \\ 1 & x_1-x_0 & & & \\ 1 & x_2-x_0 & (x_2-x_0)(x_2-x_1) & & \\ - & - & - & & \\ 1 & x_n-x_0 & (x_n-x_0)(x_n-x_1) & \cdots & (x_n-x_0)\ldots(x_n-x_{n-1}) \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} f(x_0) \\ f(x_1) \\ f(x_2) \\ \vdots \\ f(x_n) \end{pmatrix}$$