# Lecture 25

(Cont'd)

Ex $\quad f(x) = \dfrac{1}{1+25x^2}$

$$f^{(n+1)}(x) = \frac{1}{2}(-1)^{n+1}(n+1)!\left(\frac{(-5i)^{n+1}}{(1-5ix)^{n+2}} + \frac{(5i)^{n+1}}{(1+5ix)^{n+2}}\right)$$

If $\xi = 0 \Rightarrow \left|\dfrac{f^{(n+1)}(0)}{(n+1)!}\right| \sim 5^{n+1}$

If $\xi = 1 \Rightarrow \left|\dfrac{f^{(n+1)}(1)}{(n+1)!}\right| \sim \left(\dfrac{25}{26}\right)^{n+1}$

Uniform points: $\quad x_i = -1 + \dfrac{2i}{n}, \quad i = 0, \dots, n$

Chebyshev points: $\quad x_i = -\cos\left(\dfrac{\pi i}{n}\right), \quad i = 0, \dots, n$

code (matlab)

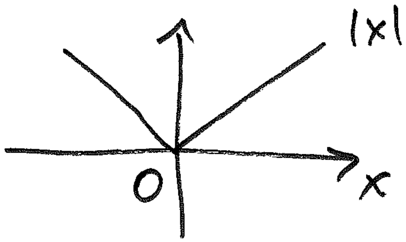
```
n=4
x=-1+2*(0:n)/n
f=ones(x)./(1+25*x.*x)
a=polyfit(x, f, n)

x=-2:0.01:2
f=ones(x)./(1+25*x.*x)
p=polyval(a,x)
plot(x,f)                or    plot(x, f, x, p,'--')
hold on
plot(x,p,'--')
```

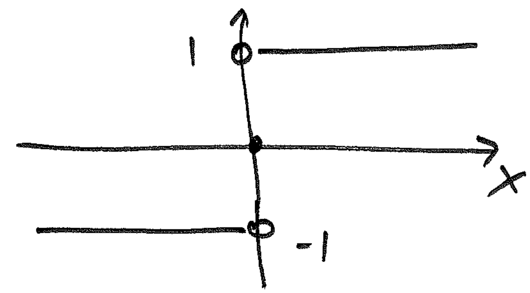
## HW

$n = 4, 8, 16, 32$

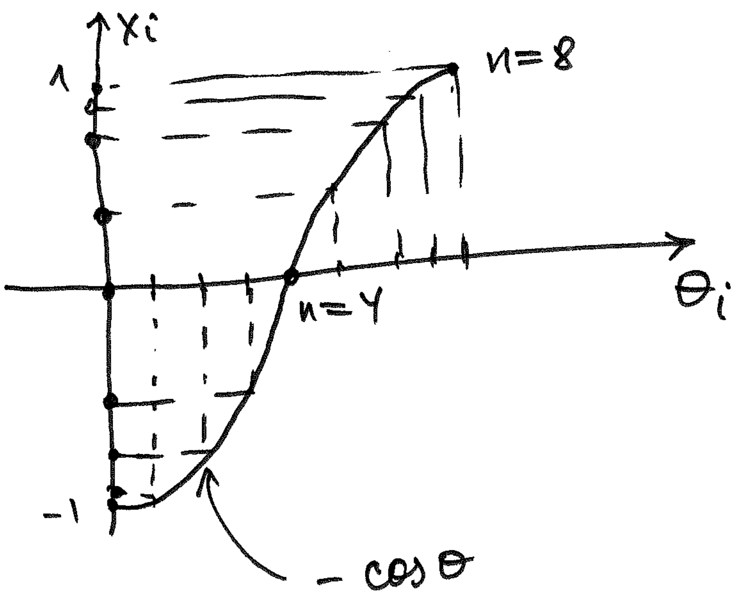
$f_1(x) = |x| = abs(x)$



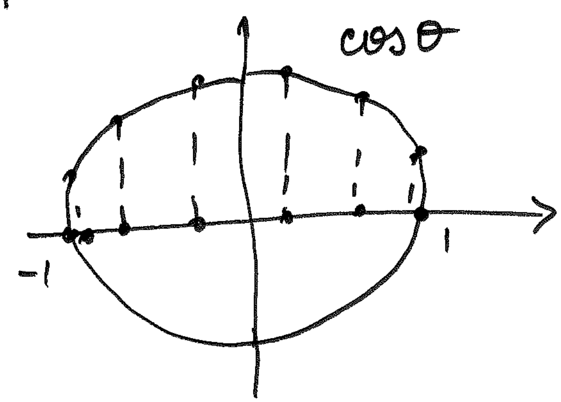

$$f_2(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$$




$x_i = -\cos\left(\frac{\pi i}{n}\right), \quad i = 0, 1, .., n$

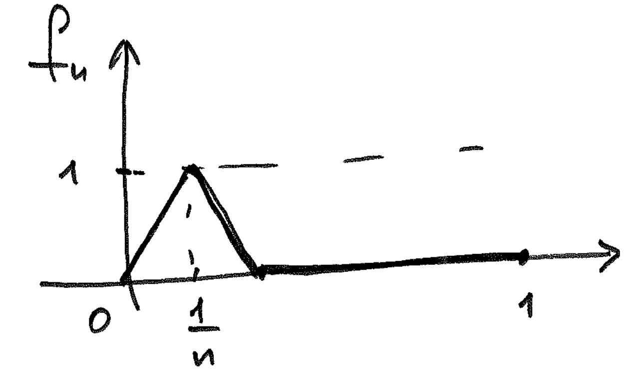



$\theta_i = \frac{\pi i}{n}$



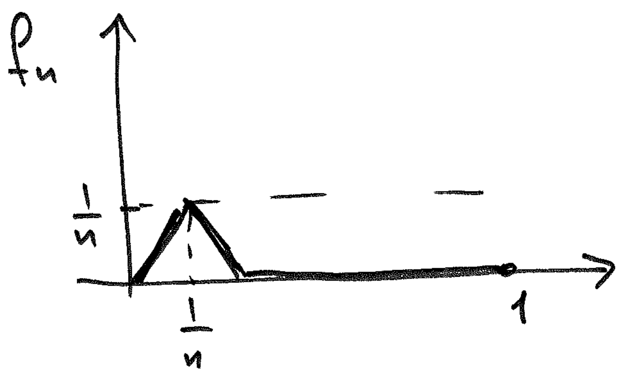
## Pointwise / uniform convergence

**Ex**




$f_n \to 0$ pointwise on $[0,1]$

$f_n \not\rightrightarrows 0$ uniformly on $[0,1]$




$f_n \to 0$ pointwise on $[0,1]$

$f_n \rightrightarrows 0$ uniformly on $[0,1]$

1. $f_n(x) \to f(x)$ pointwise on $[0,1]$

$\Leftrightarrow \lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in [0,1]$

$\Leftrightarrow$ Given $x \in [0,1]$. For any $\varepsilon > 0$, there exists $N$ such that $|f_n(x) - f(x)| < \varepsilon$ for all $n > N$, $N = (x, \varepsilon)$

2. $f_n(x) \rightrightarrows f(x)$ uniformly on $[0,1]$ $\Leftrightarrow$

$\Leftrightarrow \lim\limits_{n\to\infty} \max\limits_{0\le x\le 1} \{|f_n(x) - f(x)|\} = 0$

$\Leftrightarrow$ For any $\varepsilon > 0$, there is $N$ such that

$|f_n(x) - f(x)| < \varepsilon$ for all $x \in [0,1]$ and all $n > N$

$N = N(\varepsilon)$

**Note**

If $f_n(x) \rightrightarrows f(x)$ uniformly on $[a,b]$, then $f_n(x) \to f(x)$ pointwise on $[a,b]$

$$f_n \rightrightarrows f(x) \quad \Rightarrow \quad f_n \to f(x)$$

$$\not\Leftarrow$$

**Note**

In general, if $f(x)$ and $f'(x)$ are continuous on $[-1,1]$, then interpolating polynomial $p_n(x)$ based on Chebyshev points (Chebyshev interpolating polynomials) approach $f(x)$ uniformly on $[-1,1]$.

## Hermite interpolation

**Problem**

Given function $f(x)$, $x_0, x_1, \dots, x_n$ : $n+1$ distinct points, find a polynomial $p$ that interpolates $f$ and $f'$ at the given points, i.e.

$$p(x_i) = f(x_i) \qquad i = 0, 1, \dots, n$$
$$p'(x_i) = f'(x_i)$$

$p(x)$ is called <u>Hermite interpolating polynomial</u>,

$\deg p \le 2n+1$

Define $h_i$, $\tilde{h}_i$ polynomials of degree $\le 2n+1$ such that

$$h_i(x_j) = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases} \qquad \tilde{h}_i(x_j) = 0$$

$$h_i'(x_j) = 0 \qquad \tilde{h}_i'(x_j) = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

Then

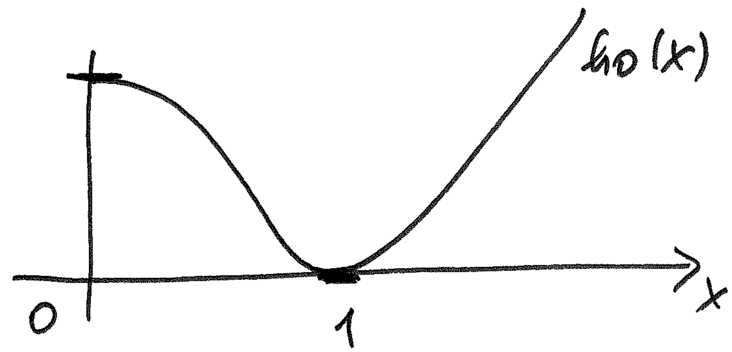
$$p(x) = \sum_{i=0}^{n} \left[ f(x_i)\, h_i(x) + f'(x_i)\, \tilde{h}_i(x) \right]$$

<u>Note</u>

This is similar to Lagrange interpolating polynomial.

**Ex** $n=1$ $\quad x_0=0,\ x_1=1 \quad 2n+1=3$

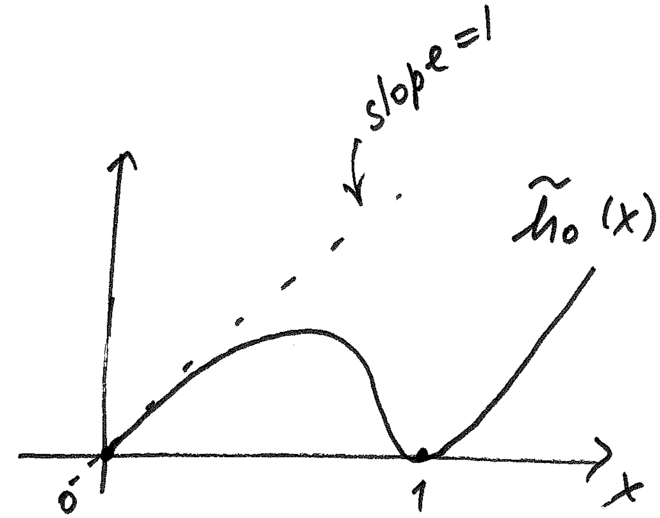
$$\left.\begin{aligned} h_0(0)&=1\\ h_0(1)&=0\\ h_0'(0)&=0\\ h_0'(1)&=0\end{aligned}\right\} \Leftrightarrow h_i(x_j)=\begin{cases}0, & i\neq j\\ 1, & i=j\end{cases} \quad \& \quad h_i'(x_j)=0$$




$\Downarrow$

$$h_0(x)=(x-1)^2(2x+1)$$

$$\begin{aligned}\tilde{h}_0(0)&=0\\ \tilde{h}_0(1)&=0\\ \tilde{h}_0'(0)&=1\\ \tilde{h}_0'(1)&=0\end{aligned} \Rightarrow \tilde{h}_0(x)=x(x-1)^2$$




Clearly, we need a more systematic way to construct $h_i$ and $\tilde{h}_i$.