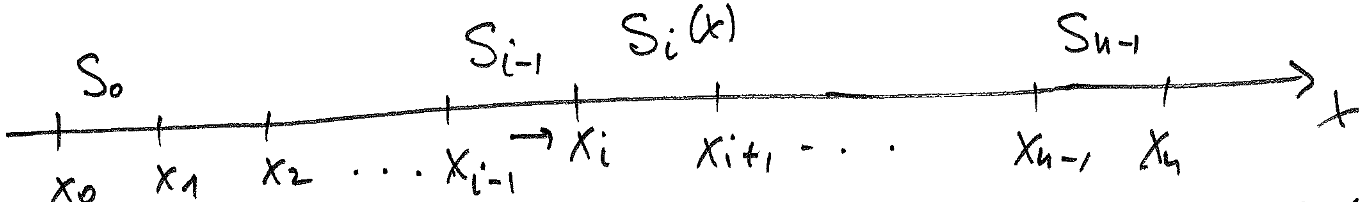
Lecture 27

Splines

Let $a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b$.

a spline of degree m is a function $S(x)$ that satisfies the following conditions:

1. For $x \in [x_i, x_{i+1}]$, $S(x) = S_i(x)$ a polynomial of degree $\le m$.

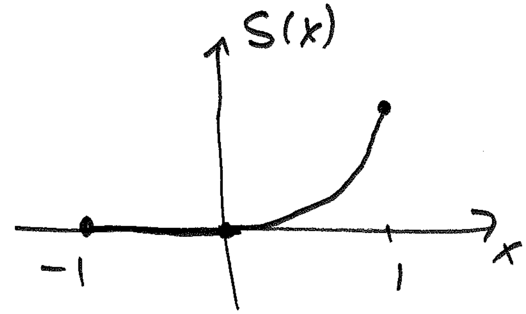


2. $S^{(m-1)}(x)$ exist and continuous at interior points $x_1, \ldots, x_{n-1}$, i.e.

$$\lim_{x \to x_i^-} S_{i-1}^{(m-1)}(x) = \lim_{x \to x_i^+} S_i^{(m-1)}(x)$$

Ex $x_0 = -1, \quad x_1 = 0, \quad x_2 = 1$

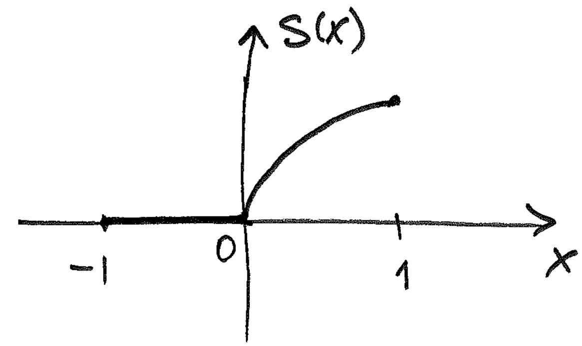
$$S(x) = \begin{cases} 0, & -1 \le x \le 0 \\ x^2, & 0 \le x \le 1 \end{cases}$$



Yes, $S(x)$ **is** a spline of degree $m=2$ (quadratic spline)

Ex

$$S(x) = \begin{cases} 0, & -1 \le x \le 0 \\ 1-(x-1)^2, & 0 \le x \le 1 \end{cases}$$



$$\lim_{x\to 0+} S'(x) = \lim_{x \to 0+} (-2)(x-1) = 2 \ne 0 = \lim_{x\to 0-} S'(x)$$

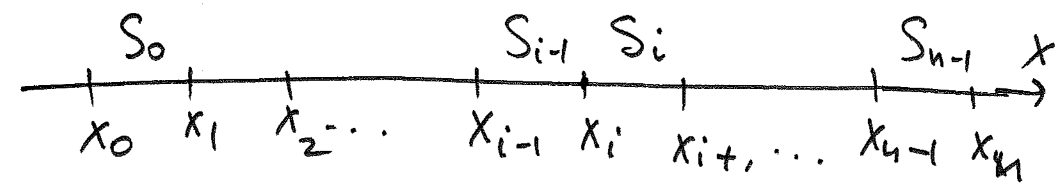
No, $S(x)$ is NOT a quadratic spline since $\lim_{x\to 0-} S'(x) \ne \lim_{x\to 0+} S'(x)$

Cubic spline interpolation

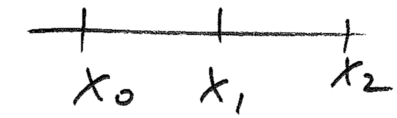
Given f, $x_0, x_1, \ldots, x_n$ as above, find a cubic spline $S(x)$ that interpolates function $f(x)$: $f(x_i) = S(x_i)$, $i=0,\ldots,n$.

$n+1$ points $\Rightarrow$ n subintervals

$\Rightarrow 4n$ coefficients



$2n = 2(n-1) + 2$ conditions to interpolate f



$2(n-1)$ conditions to require that $S'(x)$ and $S''(x)$ are continuous at interior points $x_1, x_2, \ldots, x_{n-1}$

$\Rightarrow$ we have $4n-2$ conditions and $4n$ unknowns

$\Rightarrow$ we need two extra conditions

a popular choice is

$S''(x_0) = S''(x_n) = 0$: <u>natural cubic spline interpolant</u>

Another choice:

$S'(x_0) = f'(x_0)$, $S'(x_n) = f'(x_n)$: <u>clamped cubic spline interpolation</u>

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Given $f(x)$, $x_0, x_1, \dots, x_n$: distinct points. How to construct a cubic spline $S(x)$?

Ex $x_i = ih$, $h = \frac{1}{n}$, $x_0 = 0$, $x_n = 1$

Step 1 (2^{nd} derivative condition)

S_{i-1}, S_i on x_{i-1}, x_i, x_{i+1}

We enforce: $S_{i-1}''(x_i) = S_i''(x_i)$

Since $S_i(x)$ is a cubic polynomial, $S_i''(x)$ is a linear function in x

$$\boxed{S_i''(x) = a_i \frac{x_{i+1} - x}{h} + a_{i+1} \frac{x - x_i}{h}, \quad i = 0, \dots, n-1} \qquad (*)$$

Lagrange interpol. polynomial of degree 1

Aside

$x_i \xleftrightarrow{h} x_{i+1}$

x_0, x_1

$n=1$ x_0 x_1

$$p_1 = f(x_0)\, l_0(x) + f(x_1)\, l_1(x)$$

$$l_0(x) = \frac{x - x_1}{x_0 - x_1} \qquad l_1(x) = \frac{x - x_0}{x_1 - x_0}$$

$$\Rightarrow \ell_0(x) = \frac{x - x_{i+1}}{-h}, \quad \ell_1(x) = \frac{x - x_i}{h}$$

Then

$$S_i''(x_i) = a_i \frac{x_{i+1} - x_i}{h} + a_{i+1} \frac{\cancelto{0}{x_i - x_i}}{h} = a_i \Rightarrow \boxed{S_i''(x_i) = a_i}$$

$$S_i''(x_{i+1}) = a_i \cdot \frac{\cancelto{0}{x_{i+1} - x_{i+1}}}{h} + a_{i+1} \frac{x_{i+1} - x_i}{h} = a_{i+1}$$

$$i \to i-1 \Rightarrow S_{i-1}''(x_i) = a_i$$

$$\Rightarrow \boxed{S_i''(x_i) = a_i = S_{i-1}''(x_i)}$$

Thus, $S''(x)$ is continuous at interior points $x_1, \ldots, x_{n-1}$.

Step 2 (function values)

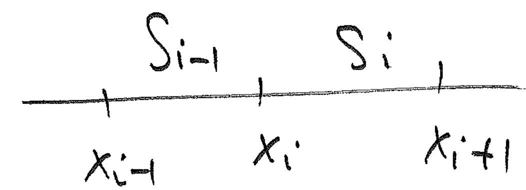
Integrate twice

unknowns

$$S_i(x) = \frac{a_i (x_{i+1} - x)^3}{6h} + \frac{a_{i+1} (x - x_i)^3}{6h} + \underbrace{b_i (x_{i+1} - x) + c_i (x - x_i)}_{\text{for convenience}}$$

$$S_i(x_i) = \frac{a_i h^2}{6} + b_i h = f(x_i) \equiv f_i$$

$$S_i(x_{i+1}) = \frac{a_{i+1} h^2}{6} + c_i h = f(x_{i+1}) \equiv f_{i+1}$$



$\Rightarrow$ we can solve for b_i and c_i:

$$b_i = \frac{f_i}{h} - \frac{a_i h}{6}, \qquad c_i = \frac{f_{i+1}}{h} - \frac{a_{i+1} h}{6}$$

Substitute:

$$S_i(x) = \frac{a_i (x_{i+1} - x)^3}{6h} + \frac{a_{i+1} (x - x_i)^3}{6h} + \left(\frac{f_i}{h} - \frac{a_i h}{6}\right)(x_{i+1} - x) + \left(\frac{f_{i+1}}{h} - \frac{a_{i+1} h}{6}\right)(x - x_i)$$

Step 3 (1st derivative condition)

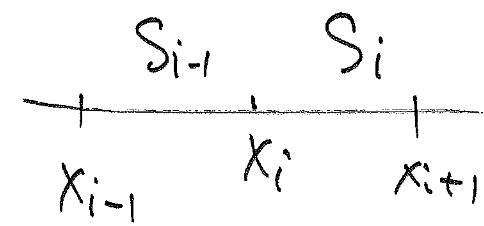
$$S_i'(x) = -\frac{a_i (x_{i+1}-x)^2}{2h} + \frac{a_{i+1}(x-x_i)^2}{2h} - \left(\frac{f_i}{h} - \frac{a_i h}{6}\right) + \left(\frac{f_{i+1}}{h} - \frac{a_{i+1} h}{6}\right)$$

$$S_i'(x_i) = -\frac{a_i h}{2} - \left(\frac{f_i}{h} - \frac{a_i h}{6}\right) + \left(\frac{f_{i+1}}{h} - \frac{a_{i+1} h}{6}\right)$$

$$S_i'(x_{i+1}) = \frac{a_{i+1} h}{2} - \left(\frac{f_i}{h} - \frac{a_i h}{6}\right) + \left(\frac{f_{i+1}}{h} - \frac{a_{i+1} h}{6}\right)$$

$i \to i-1$

$$\Rightarrow S_{i-1}'(x_i) = \frac{a_i h}{2} - \left(\frac{f_{i-1}}{h} - \frac{a_{i-1} h}{6}\right) + \left(\frac{f_i}{h} - \frac{a_i h}{6}\right)$$



We require: $S_{i-1}'(x_i) = S_i'(x_i)$

$$-\frac{a_i h}{2} - \frac{f_i}{h} + \frac{a_i h}{6} + \frac{f_{i+1}}{h} - \frac{a_{i+1} h}{6} =$$

$$= \frac{a_i h}{2} - \frac{f_{i-1}}{h} + \frac{a_{i-1} h}{6} + \frac{f_i}{h} - \frac{a_i h}{6}$$

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Simplify to get

$$a_{i-1}\frac{h}{6} + a_i\left(\frac{h}{2} - \frac{h}{6} + \frac{h}{2} - \frac{h}{6}\right) + a_{i+1}\frac{h}{6} = (f_{i-1} - 2f_i + f_{i+1})/h$$

$$/ \cdot \frac{h}{6}$$

$$\boxed{a_{i-1} + 4a_i + a_{i+1} = \frac{6}{h^2}(f_{i-1} - 2f_i + f_{i+1})} \qquad (*)$$

This holds for $i = 1, 2, \ldots, n-1$.