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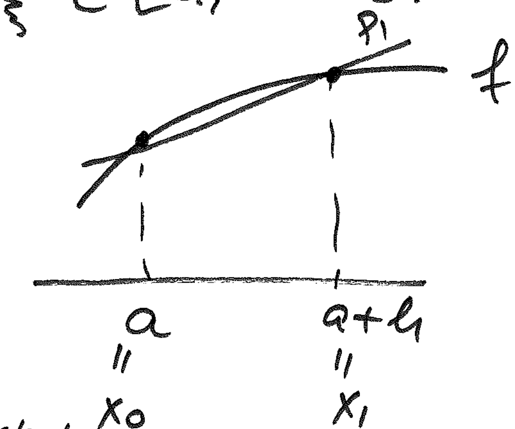
Lecture 29

Local error (Trapezoid Rule)

$$\underbrace{\int_{a}^{a+h} f(x)\,dx}_{\text{exact}} = \underbrace{h \cdot \frac{f(a)+f(a+h)}{2}}_{\text{approximation}} - \underbrace{\frac{h^3}{12} f''(\hat{\xi})}_{\text{error}}$$

x_{i+1} x_i

where $\hat{\xi}$ is between a and $a+h$, $\hat{\xi} \in [a, a+h]$.



Proof

$$f(x) = p_1(x) + \frac{f''(\xi)}{2!}(x - \underset{=a}{x_0})(x - \underset{=a+h}{x_1})$$

$p_1(x) = f[x_0] + f[x_0, x_1](x - x_0)$: linear interpolating polynomial

$$p_1(x) = f(a) + \frac{f(a+h) - f(a)}{\underset{(a+h)-a}{h}}(x-a)$$

$$\int_a^{a+h} f(x)\,dx = \int_a^{a+h} p_1(x)\,dx + \int_a^{a+h} \frac{f''(\xi)}{2!}(x-a)(x-(a+h))\,dx$$

$$\int_a^{a+h} p_1(x)\,dx = \int_a^{a+h}\left[f(a) + \frac{f(a+h)-f(a)}{h}(x-a)\right]dx = f(a)\cdot h + \frac{f(a+h)-f(a)}{h}\cdot\frac{(x-a)^2}{2}\Bigg|_{x=a}^{x=a+h} = f(a)\cdot h + \frac{f(a+h)-f(a)}{h}\cdot\frac{h^2}{2} =$$

$$= f(a)h + \left(f(a+h)-f(a)\right)\cdot\frac{h}{2} = \frac{f(a)+f(a+h)}{2}\cdot h = \frac{h}{2}\left[f(a)+f(a+h)\right]$$

In the error term, $\frac{f''(\xi)}{2!}(x-a)(x-(a+h))$, ξ is actually a function of x, i.e. changing x will change ξ. $\Rightarrow$ $\xi = \xi(x)$.

$$\int_a^{a+h} \frac{f''(\xi)}{2!}(x-a)(x-(a+h))\,dx = \left|\begin{matrix}\text{generalized}\\ \text{MVT}\end{matrix}\right| =$$

$$= \frac{f''(\hat{\xi})}{2!}\int_a^{a+h} \underbrace{(x-a)(x-(a+h))}_{<0 \text{ in } (a,\,a+h)}\,dx \; =$$

does not change its sign on $(a, a+h)$

where $\hat{\xi}$ is between a and $a+h$

$$(x-a)(x-(a+h)) = (x-x_0)(x-x_1)$$

Let $s = \frac{x-a}{h} \Rightarrow \frac{x-(a+h)}{h} = s-1$

$ds = \frac{dx}{h}$ $\quad x=a \Rightarrow s=0$

$x=a+h \Rightarrow s=1$

$$= \frac{f''(\hat{\xi})}{2!}\int_0^1 sh\cdot(s-1)h\cdot h\,ds = \frac{f''(\hat{\xi})}{2!}h^3\int_0^1 \underbrace{s(s-1)}_{s^2-s}ds =$$

$$= \frac{f''(\hat{\xi})}{2}h^3\left(\frac{s^3}{3}-\frac{s^2}{2}\right)\Big|_0^1 = -\frac{f''(\hat{\xi})}{12}h^3 \quad \blacksquare$$

Generalized MVT (Mean Value Theorem)

1. If $f(x)$ and $g(x)$ are continuous on $[a,b]$, $g(x)\geq 0$, then there exists $\xi\in[a,b]$ such that

$$\int_a^b f(x)g(x)\,dx = f(\xi)\int_a^b g(x)\,dx$$

2. Let $f(x)$ be continuous and $x_1, \dots, x_n \in [a,b]$. Then there exists $\xi \in [a,b]$ such that

$$\sum_{i=1}^{n} f(x_i) = n f(\xi) \quad \text{or}$$

$$f(\xi) = \frac{f(x_1) + \dots + f(x_n)}{n}$$

<u>Pf of 1.</u>

Denote by x_m and x_M the values at which $f(x)$ attains its min and max values, i.e.

$$f(x_m) = \min\{f(x),\ x \in [a,b]\}$$

$$f(x_M) = \max\{f(x),\ x \in [a,b]\}$$

Note $x_m, x_M \in [a,b]$ (since f is continuous on $[a,b]$)

Define $\quad h(x) = f(x) \int_a^b g(x)\,dx$

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Then since $\quad f(x_m) \le f(x) \le f(x_M) \quad \Big|\; g(x)$

$$f(x_m)\,g(x) \le f(x)\,g(x) \le f(x_M)\,g(x)$$

Integrate $\int_a^b \ldots$

$$\int_a^b f(x_m)\,g(x)\,dx \le \int_a^b f(x)\,g(x)\,dx \le \int_a^b f(x_M)\,g(x)\,dx$$

$$\underbrace{f(x_m)\int_a^b g(x)\,dx}_{h(x_m)} \le \int_a^b f(x)\,g(x)\,dx \le \underbrace{f(x_M)\int_a^b g(x)\,dx}_{h(x_M)}$$

By intermediate Value Thm, there is $\xi \in [a, b]$ such that $h(\xi) = \int_a^b f(x)\,g(x)\,dx$

but $h(\xi) = f(\xi)\int_a^b g(x)\,dx$ $\blacksquare$

2. We can write

$$\min\{n f(x),\ a \le x \le b\} \le \sum_{i=1}^{n} f(x_i) \le \max\{n f(x),\ a \le x \le b\}$$

Let $g(x) = n f(x)$

by Intermediate Value thm, there is $\xi \in [a, b]$ such that

$$\underbrace{n f(\xi)}_{g(\xi)} = \sum_{i=1}^{n} f(x_i)$$

$\blacksquare$

<u>Global error estimate</u> (Trapezoid Rule)

$$\int_a^b f(x)\,dx = T(h) - \frac{f''(\hat{\xi})}{12} h^2 (b-a), \qquad \hat{\xi} \in [a, b]$$

$$T(h) = h\left(\frac{1}{2} f(x_0) + f(x_1) + \dots + f(x_{n-1}) + \frac{1}{2} f(x_n)\right)$$

$$x_i = a + ih, \qquad h = \frac{b-a}{n}$$

Pf

$$\int_a^b f(x)\,dx = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} f(x)\,dx = \sum_{i=0}^{n-1} \left(h \frac{f(x_i) + f(x_{i+1})}{2} - \frac{h^3}{12} f''(\xi_i) \right)$$

$$= T(h) - \frac{h^3}{12} \sum_{i=0}^{n-1} f''(\xi_i) \underset{\text{MVT (2)}}{\overset{\text{generalized}}{=}} T(h) - \frac{h^3}{12} \cdot \underset{\frac{b-a}{h}}{\underset{\parallel}{n}}\, f''(\hat{\xi}) =$$

$$h = \frac{b-a}{n} \Rightarrow n = \frac{b-a}{h}$$

$$= T(h) - \frac{h^2}{12}(b-a)\, f''(\bar{\xi}) :$$ Trapezoid Rule is 2^{nd} order accurate

alternative derivation (Method of undetermined coefficients)

$$\int_0^h f(x)\,dx \sim c_0 f(0) + c_1 f(h)$$

Find values c_0 and c_1 such that integration formula is exact for polynomials of degree ≤ 1 (degree 0 and degree 1)

$f(x) = 1$ $\qquad \int_0^h 1\,dx = \boxed{h = c_0 + c_1}$

$f(x) = x$ $\qquad \int_0^h x\,dx = \frac{x^2}{2}\Big|_0^h = \boxed{\frac{h^2}{2} = c_0 \cdot \underbrace{\cancel{0}}_{f(0)} + c_1 \cdot \underbrace{h}_{f(h)}}$

$$\Rightarrow \begin{cases} c_0 + c_1 = h \\ c_1 \cdot h = \frac{h^2}{2} \end{cases} \Rightarrow c_1 = \frac{h}{2} \quad \Rightarrow c_0 = h - c_1 = h - \frac{h}{2} = \frac{h}{2}$$

$$\Rightarrow c_0 = c_1 = \frac{h}{2}$$

$$\Rightarrow \int_0^h f(x)\,dx \simeq \frac{h}{2} f(0) + \frac{h}{2} f(h) = \frac{h}{2}\left(f(0) + f(h)\right)$$