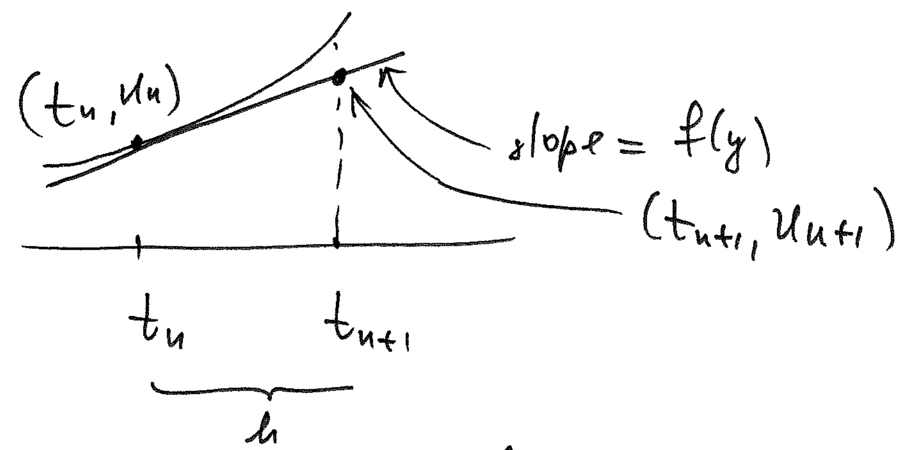
Lecture 34

Euler's method

$y' = f(y)$



We derived finite difference formula

$$\boxed{u_{n+1} = u_n + h\, f(u_n), \qquad u_0 = y_0}$$

$y(t_n)$: exact solution

$u_n \approx y(t_n)$

u_n approximation

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$\frac{dy}{dt} = f(y)$, $\frac{dy}{dt} = y'$

$\underline{\underline{\text{Ex}}}$ $y' = y$, $y(0) = 1 = y_0$

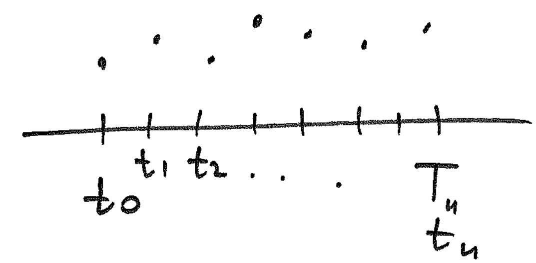
The exact solution is $y(t) = e^t$.

In this case $f(y) = y$.

Suppose we want to compute approximation of the solution at $t=1$. $y(1) = e^1 = 2.7182818$

$u_{n+1} = u_n + h f(u_n)$, $u_0 = y_0$

Choose any $n \geq 1$, then $h = \frac{1-0}{n} = \frac{1}{n}$



$u_0 = 1$ since $y(0) = 1$

$u_1 = u_0 + h \underbrace{f(u_0)}_{u_0} = u_0 + h u_0 = \underbrace{u_0}_{=1}(1+h) = 1+h$

$u_2 = u_1 + h \underbrace{f(u_1)}_{u_1} = u_1 + h u_1 = u_1(1+h) = (1+h)^2$

— — —

$u_n = (1+h)^n$

n	h	u_n (approx. at $t=1$)	$y(1) - u_n$ (global error)	$(y(1) - u_n)/h$
10	0.1	2.5937425	0.1245	1.245
20	0.05	2.6532977	0.0650	1.300
40	0.025	2.6850638	0.0332	1.328
80	0.0125	2.7014849	0.0168	1.344

Note The table indicates that

1. $\lim_{n\to\infty} u_n = y(1)$, so that the method converges
2. $\underbrace{u_n}_{\text{approx.}} = \underbrace{y(1)}_{\text{exact}} + O(h)$, i.e. the method is 1st order accurate

Code

```
h=0.25;
yo=1
tfinal = 2
nmax = tfinal/h
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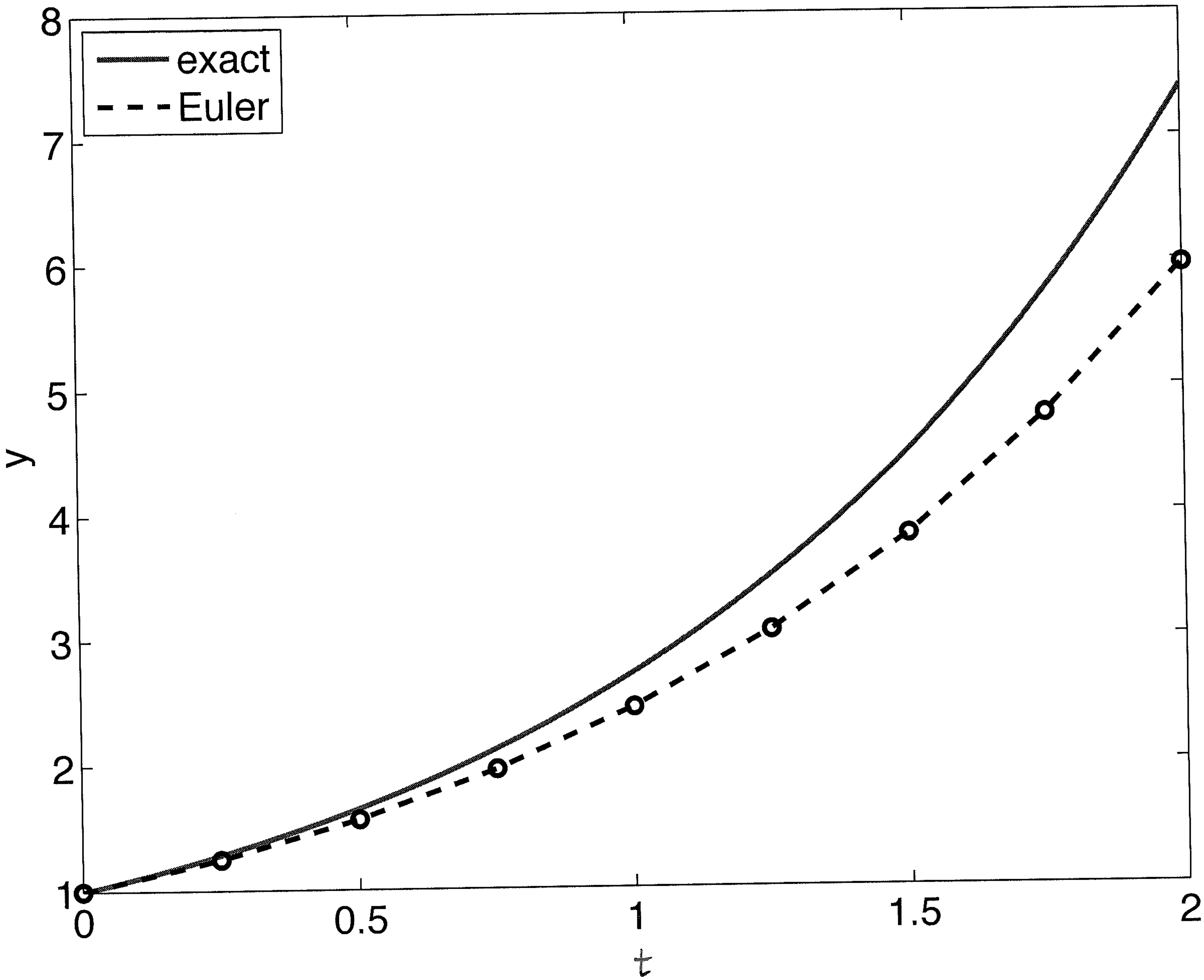
exact
Euler
y
t
0
0.5
1
1.5
2
1
2
3
4
5
6
7
8

$t = 0 : 0.01 : tfinal$

$u(1) = y_0$

for $n = 1 : nmax$

$\quad u(n+1) = u(n) + h * \underbrace{f(u(n))}_{u(n)}$

end

$tn = 0 : h : tfinal$

plot(t, y, tn, u, '--', tn, u, 'o')

Convergence (for Euler's method, special case)

$y' = y, \quad y(0) = 1, \quad y(1) = e$

$h = \frac{1-0}{n} = \frac{1}{n}$

$u_0 = 1$

$u_{n+1} = u_n + h \underbrace{f(u_n)}_{u_n} = u_n + h u_n = (1+h) u_n$

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$$\Rightarrow \quad u_n = (1+h)^n$$

$$\lim_{n\to\infty} u_n \stackrel{?}{=} y(1) = e$$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} (1+h)^n = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = 1^\infty$$

$$= e^{\ln \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n} = e^{\underbrace{\lim_{n\to\infty} \ln \left(1+\frac{1}{n}\right)^n}_{=1}} = e^1 = \underset{\underset{y(1)}{\parallel}}{e}$$

$$\begin{cases} e^{\ln t} = t \\ \ln a^n = n \ln a \end{cases}$$

$$\lim_{n\to\infty} \ln\left(1+\frac{1}{n}\right)^n = \lim_{n\to\infty} n \cdot \ln\left(1+\frac{1}{n}\right) = \infty \cdot 0 =$$

$$= \lim_{n\to\infty} \frac{\ln\left(1+\frac{1}{n}\right)}{\frac{1}{n}} = \frac{0}{0} = \lim_{n\to\infty} \frac{\frac{1}{1+\frac{1}{n}} \left(-\cancel{\frac{1}{n^2}}\right)}{-\cancel{\frac{1}{n^2}}} = 1$$

l'Hopital's rule

$$\Rightarrow \lim_{n\to\infty} \ln\left(1+\frac{1}{n}\right)^n = 1 \quad \Rightarrow \quad \lim_{n\to\infty} u_n = e^1 = e = y(1)$$

The local truncation error (local discretization error) is the amount by which the exact solution fails to satisfy the difference scheme.

Ex

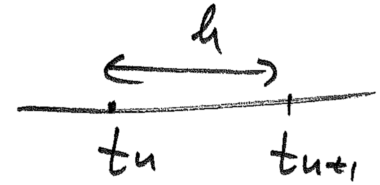
$$y' = f(y), \quad y(0) = y_0 \quad \Rightarrow \quad y(t) : \text{ exact solution}$$

$$u_{n+1} = u_n + t\, f(u_n), \quad u_0 = y_0 \; : \quad \text{Euler's method}$$

Let $y_n = y(t_n)$, where $t = t_n = nh$

$$y_{n+1} = y_n + h f(y_n) + r_n$$

where r_n is a local truncation error (error over one step)



Taylor expansion

$$y_{n+1} = y(t_{n+1}) = y(t_n + h) \underset{\substack{\text{Thm about} \\ t = t_n}}{\overset{\text{Taylor}}{=}} y(t_n) + y'(t_n) \cdot h + \frac{y''(\xi)}{2!} h^2$$

where ξ is between t_n and $t_n + h$

$$y_{n+1} = \underline{y_n} + h \cdot \underline{\underline{f(y_n)}} + r_n = \underline{y(t_n)} + \underline{\underline{y'(t_n)}} \cdot h + \frac{y''(\xi)}{2!} h^2$$

$$y' = f(y)$$

$$f(y_n) = y'(t_n) \quad \text{since} \quad y' = f(y)$$

Therefore,

$$r_n = \frac{h^2}{2} y''(\xi)$$

If $y''(\xi)$ is bounded, i.e. there is $M > 0$: $|y''(\xi)| \le M$

then

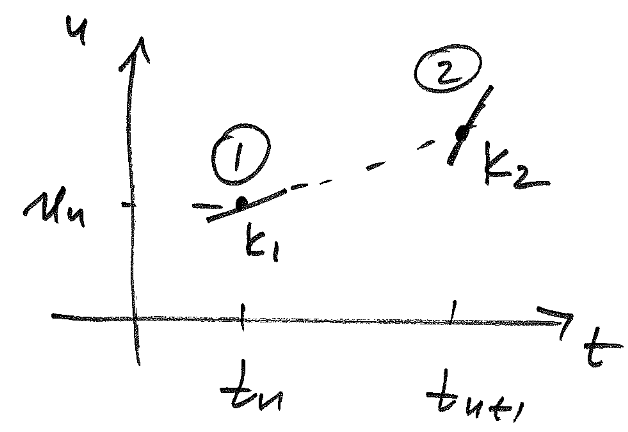
$$|r_n| = \left| \frac{h^2}{2} y''(\xi) \right| \le \frac{h^2}{2} \cdot |y''(\xi)| \le \frac{h^2}{2} \cdot M = \underbrace{\frac{M}{2}}_{C} \cdot h^2$$

$$= C h^2 = O(h^2)$$

<u>Conclusion</u>

The local truncation error for Euler's method is $O(h^2)$.

Modified Euler Method



$y' = f(y), \quad y(0) = y_0$

$$\begin{cases} k_1 = f(u_n) \\ k_2 = f(u_n + h k_1) \\ u_{n+1} = u_n + \frac{h}{2}(k_1 + k_2) \end{cases}$$

Runge-Kutta approach

$$\begin{cases} \tilde{u}_{n+1} = u_n + h f(u_n) : \quad \text{predictor (Euler)} \\ u_{n+1} = u_n + \frac{h}{2}\left(f(u_n) + f(\tilde{u}_{n+1}) \right) : \quad \text{corrector} \end{cases}$$

predictor-corrector approach

Note

1. This is an example of 2 stage Runge-Kutta method
2. Each step of modified Euler method requires two function evaluations.