Lecture 35

$\underline{\underline{\text{Ex}}}$ $y' = \underbrace{y}_{f(y)}$, $y(0) = 1$, $y(1) = 2.7182818\ldots$

$$y' = \frac{dy}{dt} = \underbrace{f(t,y)}_{\substack{\text{slope at} \\ (t,y)}}$$

$f(y) = y$

$$k_1 = f(u_n) = u_n$$

$$k_2 = f(u_n + hk_1) = u_n + hk_1 = u_n + hu_n =$$

$$= (1+h)u_n$$

$$u_{n+1} = u_n + \frac{h}{2}(k_1 + k_2) = u_n + \frac{h}{2}\Big(u_n + (1+h)u_n\Big) = u_n\left(1 + h + \frac{h^2}{2}\right)$$

$$u_{n+1} = u_n\left(1 + h + \frac{h^2}{2}\right), \quad h = \frac{1}{n}$$

n	h	u_n	$y(1) - u_n$	$(y(1) - u_n)/h^2$
10	0.1	2.71408085	0.00420098	0.4201
20	0.05	2.71719105	0.00109077	0.4363
40	0.025	2.71800394	0.00027788	0.4446
80	0.0125	2.71821170	0.00007013	0.4488

Note

The table indicates that

1. modified Euler's method converges
2. Global error / cumulative error is of 2^{nd} order, ie.

$$u_n = y(1) + O(h^2)$$

ie. modified Euler's method is 2^{nd} order accurate.

Claim

The local truncation error for modified Euler's method is $O(h^3)$.

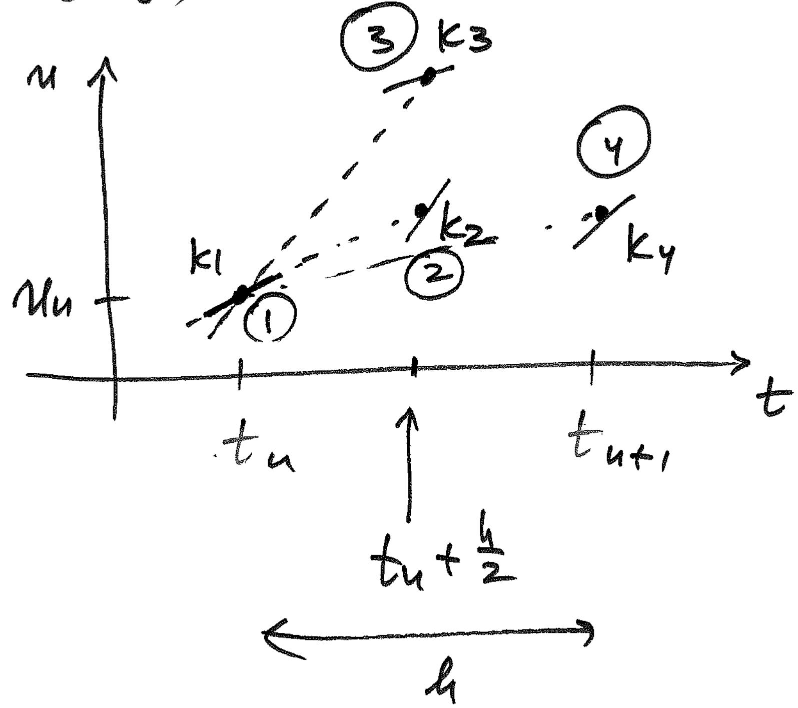
Note

The significance of the local truncation error is that if $r_n = O(h^{p+1})$, then the global error (cumulative error) is $O(h^p)$, i.e. $|u_n - y_n| = O(h^p)$.

4th order Runge–Kutta method

$$y' = f(y)$$

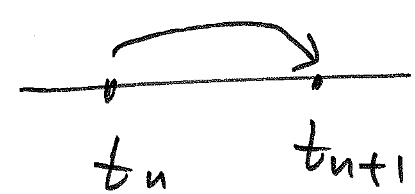
$$\begin{cases} k_1 = f(u_n) \\ k_2 = f\left(u_n + \frac{h}{2} k_1\right) \\ k_3 = f\left(u_n + \frac{h}{2} k_2\right) \\ k_4 = f(u_n + h k_3) \\ u_{n+1} = u_n + \frac{h}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right) \end{cases}$$



$$y' = f(t, y)$$

$$k_1 = f(t_n, u_n)$$

$$k_2 = f\left(t_n + \frac{h}{2}, u_n + \frac{h}{2} k_1\right)$$



$$k_3 = f\left(t_n + \frac{h}{2},\ u_n + \frac{h}{2} k_2\right)$$

$$k_4 = f(\underbrace{t_n + h}_{t_{n+1}},\ u_n + h \cdot k_3)$$

$$u_{n+1} = u_n + \frac{h}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

Note

1. The method requires 4 function evaluations per time step.
2. The local truncation error is $r_n = O(h^5)$.
3. The method is 4^{th} order accurate.

Ex $y' = y$, $y(0) = 1$, $y(1) = 2.7182818\ldots$

$k_1 = f(u_n) = \boxed{u_n}$

$f(y) = y$

$$k_2 = f\left(u_n + \frac{h}{2}k_1\right) = u_n + \frac{h}{2}k_1 = u_n + \frac{h}{2}u_n = \boxed{\left(1+\frac{h}{2}\right)u_n}$$

$$k_3 = f\left(u_n + \frac{h}{2}k_2\right) = u_n + \frac{h}{2}k_2 = u_n + \frac{h}{2}\left(1+\frac{h}{2}\right)u_n = \boxed{\left(1+\frac{h}{2}+\frac{h^2}{4}\right)u_n}$$

$$k_4 = f\left(u_n + hk_3\right) = u_n + hk_3 = u_n + h\left(1+\frac{h}{2}+\frac{h^2}{4}\right)u_n = \ldots$$

$$\ldots = \boxed{\left(1+h+\frac{h^2}{2}+\frac{h^3}{4}\right)u_n}$$

$$u_{n+1} = u_n + \frac{h}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right) = u_n + \frac{h}{6}\left(u_n + \left(1+\frac{h}{2}\right)u_n + \right.$$

$$\left. + \left(1+\frac{h}{2}+\frac{h^2}{4}\right)u_n + \left(1+h+\frac{h^2}{2}+\frac{h^3}{4}\right)u_n\right) = \ldots$$

$$\Rightarrow \boxed{u_{n+1} = \left(1+h+\frac{h^2}{2}+\frac{h^3}{6}+\frac{h^4}{24}\right)u_n}$$

<u>Note</u> solution at t_{n+1} depends only on solution at t_n

$\Rightarrow$ R-K is one-step method (but has 4 stages).

n	h	u_n	$y(1) - u_n$	$(y(1) - u_n)/h^4$
1	1.0	2.70833333	0.00994850	0.0099
2	0.5	2.71734619	0.00095637	0.0150
4	0.25	2.71820994	0.00007189	0.0184
8	0.125	2.71827684	0.00000498	0.0204

Test Equation

$y' = \lambda y, \qquad \lambda \in \mathbb{C}$

$$\frac{dy}{dt} = \lambda y$$

$$\frac{dy}{y} = \lambda \, dt \Rightarrow \int \frac{dy}{y} = \int \lambda \, dt$$

$$\ln|y| = \lambda t + \tilde{C} \qquad | \exp$$

$$y(t) = C e^{\lambda t}$$

IC: $y(0) = y_0 \Rightarrow \overbrace{y(0)}^{= y_0} = C e^0 \Rightarrow C = y_0$

$\Rightarrow$ solution is

$$\boxed{y(t) = y_0 e^{\lambda t}}$$

$$e^{a+b} = e^a \cdot e^b$$

$$y_{n+1} = y(t_{n+1}) = y(t_n + h) = y_0 e^{\lambda(t_n + h)} = \underbrace{y_0 e^{\lambda t_n}}_{\substack{y(t_n) \\ \parallel \\ y_n}} \cdot e^{\lambda h}$$

$$\rightarrow \boxed{y_{n+1} = e^{\lambda h} y_n}$$

The <u>amplification factor</u> of the exact equation $y' = \lambda y$ is $e^{\lambda h}$.

$$y' = \underbrace{\lambda y}_{f(y)}$$

<u>Euler's method</u>: $u_{n+1} = u_n + h \cdot \underbrace{f(u_n)}_{\lambda u_n} =$

$$= u_n + h \cdot \lambda u_n = (1 + h\lambda) u_n$$

The amplification factor of Euler's method is $1+h\lambda$.

For various numerical methods, the amplification factor is some function of $h\lambda$, ie.

$$u_{n+1} = Q(h\lambda)\, u_n$$

Claim

The numerical method is of order $p \Leftrightarrow$

$$Q(h\lambda) = e^{h\lambda} + O(h^{p+1})$$

Ex Euler's method

$$Q(h\lambda) = 1 + h\lambda$$

$$e^{h\lambda} = 1 + h\lambda + \frac{h^2\lambda^2}{2!} + O(h^3) \Rightarrow 1 + h\lambda = e^{h\lambda} + O(h^2)$$

$\Rightarrow$ Euler's method is 1st order accurate