University of Idaho

Lecture 41

Adams-Bashforth 2-step method

$$u_{n+1} = u_n + \frac{h}{2}\left(3 f(u_n) - f(u_{n-1})\right) \qquad y' = f(y)$$

or

$$\boxed{u_{n+1} = u_n + \frac{h}{2}\left(3 f(t_n, u_n) - f(t_{n-1}, u_{n-1})\right)} \qquad y' = f(t, y)$$

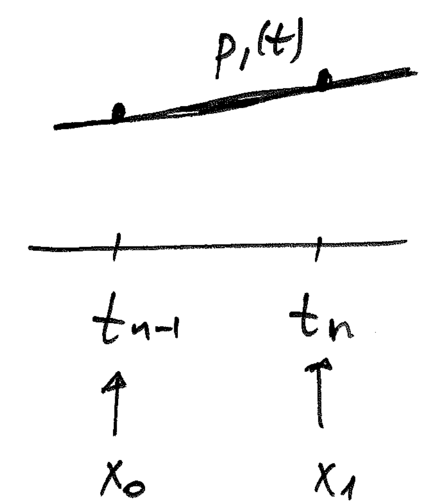
Consider

$$y'(t) = f(t, y(t)), \qquad a \le t \le b$$

$$y(a) = \alpha$$

Integrate $\int_{t_n}^{t_{n+1}}$

$$\int_{t_n}^{t_{n+1}} y'(t)\, dt = \int_{t_n}^{t_{n+1}} f(t, y(t))\, dt$$



$$y(t_{n+1}) - y(t_n) = \int_{t_n}^{t_{n+1}} f(t, y(t))\, dt$$

$$f(t, y(t)) = p_1(t) + E_1(t)$$

$$p_1(t) = l_0(t) \cdot f(t_{n-1}, y_{n-1}) + l_1(t) \cdot f(t_n, y_n)$$

$$E_1(t) = \frac{\partial^2 f}{\partial \xi^2}(\xi, y(\xi)) \cdot (t - \underset{x_0}{t_{n-1}})(t - \underset{x_1}{t_n})$$

$$l_0(t) = \frac{t - t_n}{t_{n-1} - t_n} \qquad l_1(t) = \frac{t - t_{n-1}}{t_n - t_{n-1}}$$

$$\int_{t_n}^{t_{n+1}} f(t, y(t))\, dt = \int_{t_n}^{t_{n+1}} p_1(t)\, dt + \int_{t_n}^{t_{n+1}} E_1(t)\, dt$$

$$\int_{t_n}^{t_{n+1}} p_1(t)\, dt = f(t_{n-1}, y_{n-1}) \cdot \int_{t_n}^{t_{n+1}} l_0(t)\, dt + f(t_n, y_n) \int_{t_n}^{t_{n+1}} l_1(t)\, dt$$

$$\int_{t_n}^{t_{n+1}} l_0(t)\,dt = \int_{t_n}^{t_{n+1}} \frac{t-t_n}{t_{n-1}-t_n}\,dt = -\frac{h}{2} \qquad \text{where} \quad h = t_{n+1}-t_n = t_n - t_{n-1}$$

$$\int_{t_n}^{t_{n+1}} l_1(t)\,dt = \int_{t_n}^{t_{n+1}} \frac{t-t_{n-1}}{t_n-t_{n-1}}\,dt = \frac{3h}{2}$$

$$\therefore \quad \int_{t_n}^{t_{n+1}} p_1(t)\,dt = -\frac{h}{2} f(t_{n-1}, y_{n-1}) + \frac{3h}{2} f(t_n, y_n)$$

$$\int_{t_n}^{t_{n+1}} E_1(t)\,dt = \int_{t_n}^{t_{n+1}} \frac{1}{2} \frac{\partial^2 f}{\partial t^2}\left(\xi(t), y(\xi(t))\right) \cdot (t-t_{n-1})(t-t_n)\,dt =$$

$$\underset{\text{MVT}}{\overset{\text{generalized}}{=}} \frac{1}{2} \underbrace{\frac{\partial^2 f}{\partial t^2}\left(\bar{\xi}, y(\bar{\xi})\right)}_{y'''(\bar{\xi})} \underbrace{\int_{t_n}^{t_{n+1}} (t-t_{n-1})(t-t_n)\,dt}_{=\frac{5}{6}h^3} =$$

$$= \frac{5}{12} h^3 y'''(\bar{\xi})$$

Hence,

$$y_{n+1} - y_n = \frac{h}{2}\left(3f(t_n, y_n) - f(t_{n-1}, y_{n-1})\right) + \underbrace{\frac{5}{12} h^3 y'''(\bar{\xi})}_{\tau_n \text{: local truncation error}}$$

then

$$u_{n+1} - u_n = \frac{h}{2}\left(3f(t_n, u_n) - f(t_{n-1}, u_{n-1})\right) \text{ : Adams-Bashforth method}$$

$$\tau_n = \frac{5}{12} h^3 y'''(\bar{\xi}) = O(h^3) \Rightarrow \text{AB is 2nd order accurate}$$

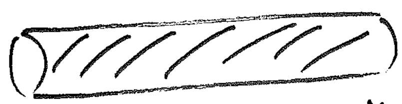
$y_n = y(t_n)$: exact solution

$u_n \approx y_n$

approximation

The linear shooting method

Application: heat flow in a thin rod



$x=a$ $\quad$ $x=b$

x : position along the rod, $\quad a \le x \le b$

t : time, $\quad 0 \le t < \infty$

$u(x,t)$: temperature at location x and time t

$f(x, u, u_x)$: external heat source

The **heat equation** from classical physics

$$u_t - u_{xx} = f$$

$u(a,t) = \alpha$, $u(b,t) = \beta$ } boundary conditions ⇒ temperature is fixed at $x=a$ and $x=b$

$u(x,0) = u_0(x)$: initial temperature

We will look for **steady-state solution**

steady-state: let $t \to \infty$, and look for solutions that are independent of time.

and independent of initial condition

$\Rightarrow u_t = 0 \quad \Rightarrow u = u(x)$

$$\frac{\partial^2 u}{\partial x^2} = u'' = \frac{d^2 u}{dx^2}$$

$$\Rightarrow \boxed{\begin{aligned} & u'' = -f \qquad a \le x \le b \\ & u(a) = \alpha \\ & u(b) = \beta \end{aligned}} \qquad (*)$$

the steady-state heat is an ODE, but it is not an IVP.

for IVP: $u(a) = \alpha, \quad u'(a) = \alpha_2$

BVP: $u(a) = \alpha, \quad u(b) = \beta$

(*) is a boundary value problem.

We considered another BVP

We used finite difference method to solve it.

We will use another approach and reduce our BVP to IVP.

I. The linear shooting method

Assume $f(x, u, u')$ is a linear function of x, u, and u'.

Most general case: $f = -p(x)u' - q(x)u - r(x)$

$\Rightarrow$ steady-state heat eq^{n}

$$u'' = p(x)u' + q(x)u + r(x), \quad a \le x \le b$$
$$u(a) = \alpha$$
$$u(b) = \beta \qquad (**)$$

Thm

If

1) $p(x)$, $q(x)$, and $r(x)$ are continuous on $[a, b]$
2) $q(x) > 0$ on $[a, b]$

then the BVP $(**)$ has a unique solution.

<u>Idea</u>: let $u(x) = v(x) + c\,w(x)$

Boundary condition at $x=a$: $u(a) = v(a) + c\,w(a)$

given: $u(a) = \alpha$ $\alpha = \alpha + c \cdot 0$

let $v(a) = \alpha$, $w(a) = 0$

Boundary condition at $x = b$: $u(b) = v(b) + c\,w(b)$

$u(b) = \beta$ $\Rightarrow$ $\beta = v(b) + c\,w(b)$

$$\Rightarrow \boxed{c = \frac{\beta - v(b)}{w(b)}}$$

$$\therefore \boxed{u(x) = v(x) + \frac{\beta - v(b)}{w(b)}\,w(x) \quad \text{where } v(a) = \alpha,\ w(a) = 0}$$

Equation: $u'' = pu' + qu + r$ $u = v + c \cdot w$

$$\underline{v''} + c\,w'' = p(\underline{v'} + c\,w') + q(\underline{v} + c\,w) + \underline{r}$$

Let $v'' = pv' + qv + r$

$w'' = pw' + qw$

Note v and w both satisfy 2^{nd} order DEs

but we only have 1 condition on each:

$v(a) = \alpha, \quad w(a) = 0$

we can add extra condition we want!

IVP: $v'' = p(x)v' + q(x)v + r(x), \quad a \le x \le b$

$v(a) = \alpha$

$v'(a) = 0$

IVP: $w'' = p(x)w' + q(x)w, \quad a \le x \le b$

$w(a) = 0$

$w'(a) = 1$

At the end: $u(x) = v(x) + \frac{\beta - v(b)}{w(b)} w(x)$

Claim $w(b) \neq 0$

Proof Consider IVP for $w(x)$

$w(b) = 0$

$\Rightarrow w'' = p(x) w' + q(x) w$

$w(a) = 0$

$w(b) = 0$

$\Rightarrow w(x) \equiv 0$ is a solution of this IVP. Since p, q are continuous, $w(x) \equiv 0$ is unique. $\Rightarrow w' \equiv 0$ but $w'(a) = 1$ ↯

$\Rightarrow w(b) \neq 0.$ ■