Thm

Let A be an $n \times n$ matrix. The following statements are equivalent.

1. The equation $Ax = b$ has a unique solution for any b.
2. $\det A \neq 0$
3. Matrix A is invertible, i.e. there is matrix that we denote by A^{-1} such that $AA^{-1} = A^{-1}A = I$

   I: identity matrix
   $$I = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ & & \ddots \\ 0 & & 1 \end{pmatrix}$$
   A^{-1}: inverse matrix
4. Rows of A are linearly independent.
5. Eigenvalues of A are all nonzero.
6. The equation $Ax = 0$ has a unique solution $x = 0$.

Direct Methods

$Ux = b$ where U is upper triangular matrix

$$\begin{pmatrix} \times & & * \\ & \times & * \\ 0 & & \times \end{pmatrix}$$

$$
\begin{aligned}
u_{11} x_1 + u_{12} x_2 + \dots &+ u_{1n} x_n = b_1 \\
u_{22} x_2 + \dots &+ u_{2n} x_n = b_2 \\
&\ddots \\
u_{n-1,n-1} x_{n-1} &+ u_{n-1,n} x_n = b_{n-1} \\
& u_{nn} x_n = b_n
\end{aligned}
$$

Then

$$x_n = b_n / u_{nn}$$

$$x_{n-1} = (b_{n-1} - u_{n-1,n} x_n) / u_{nn}$$

$$- - - - - - -$$

$$x_1 = \left(b_1 - (u_{12} x_2 + \dots + u_{1n} x_n)\right) / u_{11}$$

This is called <u>back substitution</u>.

Operation count

\# divisions $= n$

\# multiplications $= \dfrac{n(n-1)}{2}$

Proof

\# mults $= 1+2+\dots+(n-1) = \sum_{k=1}^{n-1} k = S$

$$2S = \sum_{k=1}^{n-1} k + \sum_{k=1}^{n-1} (n-k) = \sum_{k=1}^{n-1} \left(\cancel{k} + (n-\cancel{k})\right) = \sum_{k=1}^{n-1} n = n(n-1)$$

$$\Rightarrow S = \frac{1}{2} n(n-1)$$

The leading order term in operation count is $\dfrac{n^2}{2}$.

$Lx = b$ where L is lower triangular matrix $L = \begin{pmatrix} * & & 0 \\ * & * & \end{pmatrix}$

$$l_{11} x_1 = b_1$$

$$l_{21} x_1 + l_{22} x_2 = b_2$$

$$\text{— — — — —}$$

$$l_{n1} x_1 + l_{n2} x_2 + \dots + l_{nn} x_n = b_n$$

Then solve for $x_1, x_2, \dots, x_n$: <u>forward elimination</u>

The operation count is again $\sim \frac{n^2}{2}$.

<u>Gaussian Elimination</u>

<u>idea</u>: use <u>elementary operations</u> to reduce $Ax = b$ to the <u>upper triangular matrix</u> and use <u>back substitution</u> to find x.

1. multiply an equation by a nonzero constant and subtract from another equation
2. interchange two equation

Ex $\left(\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right)$ augmented matrix

Step 1 eliminate variable x_1 from eq^us 2 and 3.

$m_{21} = \dfrac{a_{21}}{a_{11}}$

$$a_{22} \leftarrow a_{22} - m_{21} a_{12}$$
$$a_{23} \leftarrow a_{23} - m_{21} \cdot a_{13}$$
$$b_2 \leftarrow b_2 - m_{21} b_1$$

$m_{31} = \dfrac{a_{31}}{a_{11}}$

$$a_{32} \leftarrow a_{32} - m_{31} a_{12}$$
$$a_{33} \leftarrow a_{33} - m_{31} a_{13}$$
$$b_3 \leftarrow b_3 - m_{31} b_1$$

$$\left(\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a_{22} & a_{23} & b_2 \\ 0 & a_{32} & a_{33} & b_3 \end{array}\right)$$ These elements have been changed.

<u>Step 2</u> Eliminate variable x_2 from equation 3.

$$m_{32} = \frac{a_{32}}{a_{22}} \qquad a_{33} \leftarrow a_{33} - m_{32}\, a_{23}$$

$$b_3 \leftarrow b_3 - m_{32}\, b_2$$

$$\left(\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a_{22} & a_{23} & b_2 \\ 0 & 0 & a_{33} & b_3 \end{array}\right)$$ These elements have been changed.

Now we can use back substitution to find solution.

<u>Ex</u>

$$\begin{aligned} 2x_1 - x_2 \quad\quad &= 1 \\ -x_1 + 2x_2 - x_3 &= 0 \\ -x_2 + 2x_3 &= 1 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 2 & -1 & 0 & 1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & 1 \end{array}\right)$$

$m_{21} = \frac{-1}{2}$

$$2 \leftarrow 2 - \left(-\tfrac{1}{2}\right)(-1) = \tfrac{3}{2}$$

$$-1 \leftarrow -1 - \left(-\tfrac{1}{2}\right)\cdot 0 = -1$$

$$0 \leftarrow 0 - \left(-\tfrac{1}{2}\right)1 = \tfrac{1}{2}$$

$m_{31} = 0$

$$\left(\begin{array}{ccc|c} 2 & -1 & 0 & 1 \\ 0 & \frac{3}{2} & -1 & \frac{1}{2} \\ 0 & -1 & 2 & 1 \end{array}\right)$$

$$m_{32} = \frac{-1}{3/2} = -\frac{2}{3}$$

$$2 \Leftarrow 2 - \left(-\frac{2}{3}\right)(-1) = \frac{4}{3}$$

$$1 \Leftarrow 1 - \left(-\frac{2}{3}\right)\cdot\frac{1}{2} = \frac{4}{3}$$

$$\left(\begin{array}{ccc|c} 2 & -1 & 0 & 1 \\ 0 & \frac{3}{2} & -1 & 1/2 \\ 0 & 0 & \frac{4}{3} & 4/3 \end{array}\right)$$

$$\begin{aligned} 2x_1 - x_2 &= 1 \\ \tfrac{3}{2}x_2 - x_3 &= \tfrac{1}{2} \\ \tfrac{4}{3}x_3 &= \tfrac{4}{3} \end{aligned} \uparrow$$

$$x_3 = \frac{4}{3} \Big/ \left(\frac{4}{3}\right) = 1$$

$$x_2 = \left(\frac{1}{2} - (-1)\cdot 1\right) \Big/ \frac{3}{2} = 1$$

$$x_1 = \left(1 - (-1\cdot 1 + 0\cdot 1)\right) / 2 = 1$$

$$\Rightarrow \quad \vec{x} = \begin{pmatrix} \mid \\ \mid \\ \mid \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

General case

$$\left(A^{(k)} : b^{(k)}\right) = \left(\begin{array}{cccccc|c} a_{11}^{(1)} & a_{12}^{(1)} & - & - & - & - & b_1^{(1)} \\ & a_{22}^{(2)} & - & - & - & - & b_2^{(2)} \\ & & \ddots & & & & \vdots \\ & & & a_{kk}^{(k)} & \cdots & a_{kn}^{(k)} & b_k^{(k)} \\ & 0 & & \vdots & & \vdots & \vdots \\ & & & a_{nk}^{(k)} & \cdots & a_{nn}^{(k)} & b_n^{(k)} \end{array}\right)$$

Step k: eliminate variable x_k from equations $k+1, .., n$

$$m_{ik} = \frac{a_{ik}^{(k)}}{a_{kk}^{(k)}}, \qquad i = k+1, .., n$$

$$a_{ij}^{(k+1)} = a_{ij}^{(k)} - m_{ik} \cdot a_{kj}^{(k)}, \qquad i,j = k+1, .., n$$

$$b_i^{(k+1)} = b_i^{(k)} - m_{ik} \cdot b_k^{(k)}, \quad i = k+1, \dots, n$$

m_{ik} : multiplier

$a_{kk}^{(k)}$: pivot

After the last step, pivot elements will be on the main diagonal.

code

```
% reduction to upper triangular form

for k=1:n-1
    for i=k+1:n
        xm = a(i,k) / a(k,k)
            for j=k+1:n
                a(i,j) = a(i,j) - xm * a(k,j)
            end
            b(i) = b(i) - xm * b(k)
    end
end
```