Lecture 9

```
% back substitution
%
x(n) = b(n) / a(n,n);
for  i = n-1 : -1 : 1      % -1 : step size
   s=0;
   for j = i+1 : n
      s = s + a(i,j) * x(j);
   end
   x(i) = (b(i) - s) / a(i,i);
end
```

$i = 1 : N \iff i = 1 : 1 : N$

$i = 1 : \underbrace{2}_{\text{step}} : N$

Note $a_{11}x_1 + \underbrace{a_{12}x_2 + \dots + a_{1n}x_n}_{S} = b_1$

Operation count (reduction to upper Δ form)

$\#\text{ divisions} = 1 + 2 + \dots + (n-1) = \frac{n(n-1)}{2}$

$\#\text{ multiplications} = n(n-1)(2n-1)/6$

Pf

$$\# \text{ divisions } = \sum_{k=1}^{n-1} (n-k) = \sum_{k=1}^{n-1} k = \frac{n(n-1)}{2}$$

$$\# \text{ mult } = \sum_{k=1}^{n-1} (n-k)^2 = \sum_{k=1}^{n-1} k^2 = S$$

$$(n-1)^3 = (n-1)^3 - (n-2)^3 + (n-2)^3 + \dots - 2^3 + 2^3 - 1^3 + 1^3 =$$

$$= \sum_{k=1}^{n-1} \left(k^3 - (k-1)^3\right) = \sum_{k=1}^{n-1} \left(\cancel{k^3} - (\cancel{k^3} - 3k^2 + 3k - 1)\right) =$$

$$= 3 \underbrace{\sum_{k=1}^{n-1} k^2}_{S} - 3 \underbrace{\sum_{k=1}^{n-1} k}_{\frac{n(n-1)}{2}} + \underbrace{\sum_{k=1}^{n-1} 1}_{n-1}$$

$$\Rightarrow (n-1)^3 = 3S - 3\frac{n(n-1)}{2} + (n-1)$$

Solve for S.

$$S = \frac{n(n-1)(2n-1)}{6} \quad \blacksquare$$

Note The leading order in operation count in Gaussian elimination is $\frac{n^3}{3}$.

Matrix Form

Let $A^{(k)}$ be the reduced matrix at step k. The effect of performing step k is

$$A^{(k+1)} = E_k A^{(k)}$$

where

$$E_k = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & 0 & \\ & & 1 & & & \\ & & -m_{k+1,k} & 1 & & \\ & 0 & \vdots & & \ddots & \\ & & -m_{nk} & & & 1 \end{pmatrix} : \text{ lower triangular matrix}$$

row $k+1$
column k

$$\begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$$

$$\text{row}(k+1) \leftarrow \text{row}(k+1) - m_{k+1,k} \cdot \text{row}(k)$$

Then $Ax = b$ is reduced to $A^{(n)}x = b^{(n)}$ where

$A^{(n)} = E_{n-1} \dots E_2 E_1 A$: upper triangular matrix

$b^{(n)} = E_{n-1} \dots E_2 E_1 b$

LU factorization

Assume $E_{n-1} \dots E_2 E_1 A = U$: upper triangular matrix ($a_{kk}^{(k)} \neq 0$)

$$E_k = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & 0 & \\ & & 1 & & & \\ & & -m_{k+1,k} & 1 & & \\ & 0 & \vdots & & \ddots & \\ & & -m_{n,k} & & & 1 \end{pmatrix} = I - m_k e_k^T$$

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where

$$m_k = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ m_{k+1,k} \\ \vdots \\ m_{nk} \end{pmatrix} \qquad e_k = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow k^{th} \text{ row}$$

$\underline{\underline{Ex}}$ 3 x 3 case

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ -m_{21} & 1 & 0 \\ -m_{31} & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ m_{21} & 0 & 0 \\ m_{31} & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ m_{21} & 0 & 0 \\ m_{31} & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 \\ m_{21} \\ m_{31} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$$

Since $\quad U = E_{n-1} \dots E_2 E_1 A$ $\qquad\qquad (AB)^{-1} = B^{-1}A^{-1}$

$$A = (E_{n-1} \dots E_2 E_1)^{-1} U$$

$$A = \underbrace{E_1^{-1} E_2^{-1} \dots E_{n-1}^{-1}}_{?} \cdot U$$

1. $E_k^{-1} = (I - m_k e_k^T)^{-1} = I + m_k e_k^T$

Proof

$$(I - m_k e_k^T)(I + m_k e_k^T) = I + \cancel{m_k e_k^T} - \cancel{m_k e_k^T} - m_k \underbrace{\cancel{e_k^T m_k}}_{0} e_k^T$$

$$e_k^T m_k = \begin{array}{c} (0 \dots 0\ 1\ 0 \dots 0) \\ \uparrow \\ \text{column } k \end{array} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ m_{k+1,k} \\ \vdots \\ m_{nk} \end{pmatrix} \begin{array}{l} \left.\begin{array}{c} \\ \\ \\ \end{array}\right\} k \text{ rows} \\ \\ \\ \end{array} = 0$$

2. $E_1^{-1} E_2^{-1} = I + m_1 e_1^T + m_2 e_2^T$

Proof

$$E_1^{-1} E_2^{-1} = (I + m_1 e_1^T)(I + m_2 e_2^T) = I + m_1 e_1^T + m_2 e_2^T + m_1 \cancel{\underbrace{e_1^T m_2}_{0}} e_2^T$$

$$e_1^T m_2 = (1 \; 0 \; \dots \; 0) \begin{pmatrix} 0 \\ 0 \\ m_{32} \\ \vdots \\ m_{n2} \end{pmatrix} = 0$$

$$E_1^{-1} E_2^{-1} = \begin{pmatrix} 1 & & & & \\ m_{21} & 1 & & & \\ m_{31} & m_{32} & 1 & & \\ \vdots & \vdots & & \ddots & \\ m_{n1} & m_{n2} & 0 & & 1 \end{pmatrix}$$

3. $$E_1^{-1} E_2^{-1} \dots E_{n-1}^{-1} = \begin{pmatrix} 1 & & & & 0 \\ m_{21} & 1 & & & \\ m_{31} & m_{32} & \ddots & & \\ \vdots & \vdots & & \ddots & \\ m_{n1} & m_{n2} & \cdots & m_{n,n-1} & 1 \end{pmatrix} =$$

$$= I + m_1 e_1^T + m_2 e_2^T + \dots + m_{n-1} e_{n-1}^T = L$$: lower triangular matrix

Then $$A = \underbrace{E_1^{-1} E_2^{-1} \dots E_{n-1}^{-1}}_{L} U \Rightarrow A = LU$$: LU factorization of A

where $L = E_1^{-1} E_2^{-1} \dots E_{n-1}^{-1}$

<u>Application</u>

Solve $Ax = b$

1. Find $L, \bar{U}$ such that $A = L\bar{U}$ $\quad O(n^3)$
2. Solve $Ly = b$ for y $\quad O(n^2)$
3. Solve $\bar{U}x = y$ for x $\quad O(n^2)$

Steps 2 and 3: triangular systems

$$Ax = b$$
$$L\underbrace{\bar{U}x}_{y} = b$$
$$\bar{U}x = y$$

Then $\quad Ax = L\underbrace{\bar{U}x}_{y} = Ly = b$

<u>Advantage</u> If you solve for L and U, then you can solve for many vectors b using steps 2 and 3.

<u>Storage</u>

Instead of storing L and $\bar{U}$, you can overwrite A with reduced A, i.e. $\bar{U}$, and entries of L:

$$L \rightarrow \begin{pmatrix} \tilde{a}_{11} & \cdots & \tilde{a}_{1n} \\ m_{21} & \ddots & \vdots \\ \vdots & & \tilde{a}_{nn} \\ m_{n1} & \cdots \; m_{n,n-1} & \end{pmatrix} \leftarrow U$$

<u>triangularization</u>