2.1. THE BINARY NUMBER SYSTEM

In the decimal system, a number such as 342.105 means

$$3 \cdot 10^2 + 4 \cdot 10^1 + 2 \cdot 10^0 + 1 \cdot 10^{-1} + 0 \cdot 10^{-2} + 5 \cdot 10^{-3} \tag{2.1}$$

Numbers written in the decimal system are interpreted as a sum of multiples of integer powers of 10. There are 10 digits, denoted by $0, 1, \ldots, 9$. We say 10 is the base of the decimal system.

The binary system represents all numbers as a sum of multiples of integer powers of 2. There are two digits, 0 and 1; and 2 is the base of the binary system. The digits 0 and 1 are called *bits*, which is short for *binary digits*. For example, the number 1101.11 in the binary system represents the number

$$1 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0 + 1 \cdot 2^{-1} + 1 \cdot 2^{-2} \tag{2.2}$$

in the decimal system. For clarity when discussing a number with respect to different bases, we will enclose the number in parentheses and give the base as a subscript. For example,

$$(1101.11)_2 = (13.75)_{10}$$

To convert a general binary number to its decimal equivalent, proceed as in (2.2).

EXAMPLE Consider $(111 \ldots 1)_2$ with n consecutive 1's to the left of the binary point. This has the decimal equivalent

$$2^{n-1} + 2^{n-2} + \cdots + 2^1 + 2^0 = 2^n - 1$$

using formula (1.16) for the sum of a geometric series. Thus

$$(11 \ldots 11)_2 = (2^n - 1)_{10} \tag{2.3}$$

where the binary number has n digits. ■

The principles behind the arithmetic operations are the same in the binary system and the decimal system, with the major difference being that in the binary system fewer digits are allowed. As examples, consider the following addition and multiplication calculations:

$$\begin{array}{r} 11110 \\ +\quad 1101 \\ \hline 101011 \end{array} \qquad\qquad \begin{array}{r} 111 \\ \times\quad 110 \\ \hline 000 \\ 111 \\ 111 \\ \hline 101010 \end{array}$$

Tables 2.1 and 2.2 contain addition and multiplication tables for the binary numbers corresponding to the decimal digits 1, 2, 3, 4, and 5.

Table 2.1. Binary Addition

+	1	10	11	100	101
1	10	11	100	101	110
10	11	100	101	110	111
11	100	101	110	111	1000
100	101	110	111	1000	1001
101	110	111	1000	1001	1010

Table 2.2. Binary Multiplication

×	1	10	11	100	101
1	1	10	11	100	101
10	10	100	110	1000	1010
11	11	110	1001	1100	1111
100	100	1000	1100	10000	10100
101	101	1010	1111	10100	11001

Conversion from Decimal to Binary

We will give methods for converting decimal integers and decimal fractions to binary integers and binary fractions. These methods will be in a form convenient for hand computation. Different algorithms are required when doing such conversions within a binary computer, but we will not give them in this text.

Suppose that x is an integer written in decimal. We want to find coefficients $a_0, a_1, \ldots, a_n$, all 0 or 1, for which

$$a_n \cdot 2^n + a_{n-1} \cdot 2^{n-1} + \cdots + a_1 \cdot 2^1 + a_0 \cdot 2^0 = x \tag{2.4}$$

The binary integer will be

$$(a_n a_{n-1} \cdots a_0)_2 = (x)_{10} \tag{2.5}$$

To find the coefficients, begin by dividing x by 2, and denote the quotient by x_1. The remainder is a_0. Next divide x_1 by 2, and denote the quotient by x_2. The remainder is a_1. Continue this process, finding $a_2, a_3, a_4, \ldots, a_n$ in succession.

EXAMPLE The following shortened form of the above method is convenient for hand computation. Convert $(19)_{10}$ to binary.

$$
\begin{array}{llllll}
2\,|\,19 & = x & & & & \\
\quad 2\,|\,9 & & = x_1 & & & a_0 = 1 \\
\quad\quad 2\,|\,4 & & & = x_2 & & a_1 = 1 \\
\quad\quad\quad 2\,|\,2 & & & = x_3 & & a_2 = 0 \\
\quad\quad\quad\quad 2\,|\,1 & & & = x_4 & & a_3 = 0 \\
\quad\quad\quad\quad\quad 0 & & & & = x_5 & a_4 = 1
\end{array}
$$

Thus $(19)_{10} = (10011)_2$. ∎

Suppose now that x is a decimal fraction and that it is positive and less than 1.0. Then we want to find coefficients $a_1, a_2, a_3, \ldots$, all 0 or 1, for which

$$a_1 \cdot 2^{-1} + a_2 \cdot 2^{-2} + a_3 \cdot 2^{-3} + \cdots = x \tag{2.6}$$

The binary fraction will be

$$(.a_1 a_2 a_3 \ldots)_2 = (x)_{10} \tag{2.7}$$

To find the coefficients, begin by denoting $x = x_1$. Multiply x_1 by 2, and denote $x_2 = \text{Frac}(2x_1)$, the fractional part of $2x_1$. The integer part $\text{Int}(2x_1)$ equals a_1. Repeat the process. Multiply x_2 by 2, letting $x_3 = \text{Frac}(2x_2)$ and $a_2 = \text{Int}(2x_2)$. Continue in the same manner, obtaining $a_3, a_4, \ldots$ in succession. ∎

EXAMPLE Find the binary form of 5.578125. We break the numbers into an integer and a fraction part. As in the previous example, we find that

$$(5)_{10} = (101)_2$$

For the fractional part $x_1 = x = 0.578125$, use the above algorithm.

$$
\begin{array}{lll}
2x_1 = 1.15625 & x_2 = 0.15625 & a_1 = 1 \\
2x_2 = 0.3125 & x_3 = 0.3125 & a_2 = 0 \\
2x_3 = 0.625 & x_4 = 0.625 & a_3 = 0 \\
2x_4 = 1.25 & x_5 = 0.25 & a_4 = 1 \\
2x_5 = 0.5 & x_6 = 0.5 & a_5 = 0 \\
2x_6 = 1.0 & x_0 = 0 & a_6 = 1
\end{array}
$$

Thus the binary equivalent is 0.100101; combining it with the earlier result, we get

$$(5.578125)_{10} = (101.100101)_2 \quad ∎$$

EXAMPLE Convert the decimal fraction 0.1 to its binary equivalent. By using the above procedure, we obtain

$$(0.1)_{10} = (0.00011001100110\ldots)_2 \tag{2.8}$$

an infinite repeating binary fraction. Numbers have a finite binary fractional form if and only if they are expressible as a sum of a finite number of negative powers of 2. The decimal number 0.1 is not expressible as such a finite sum. This result has implications when one is working with finite decimal numbers on a binary computer, and we will consider those implications later in this chapter and in the next chapter. ■