Factorial experiments

In a **factorial experiment**, a response variable Y is observed at all factor-level combinations of two or more factors.

Two factor experiment

		factor B			
		1	2	$\cdots$	b
factor A	1				
	2				
	$\vdots$				
	a				

complete factorial: at least one observation in every cell

balanced: n observations in every cell ($n = 1$: interaction between A and B cannot be estimated)

randomized: experimental units assigned to treatment combinations at random

Model

One could write the model as

$$Y_{ijk} = \mu_{ij} + \epsilon_{ijk}$$

where μ_{ij} is the mean in the (i, j)th cell; this is essentially the form of a one-way AOV model.

Better way is to exploit the two-way structure by parameterizing the means as effects due to the different factors. In the additive, or main-effects only, model, we write

$$Y_{ijk} = \mu + \alpha_i + \beta_j + \epsilon_{ijk}$$

In the interaction model, we write

$$Y_{ijk} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \epsilon_{ijk}$$

Here:

α_i is the effect of factor A, level i ($i = 1, 2, ..., a$)

β_j is the effect of factor B, level j ($j = 1, 2, ..., b$)

$(\alpha\beta)_{ij}$ is the correction in the (i, j)th cell mean due to interaction between factors A and B

$\sum_i \alpha_i = 0$ $\quad$ $\sum_j \beta_j = 0$ $\quad$ $\sum_i (\alpha\beta)_{ij} = 0$ $\quad$ $\sum_j (\alpha\beta)_{ij} = 0$

$\epsilon_{ijk} \sim \text{normal}(0, \sigma^2)$

$k = 1, 2, ..., n$ (balanced design; or n_{ij} in unbalanced)

Additivity: lines joining means for a factor (within levels of the other factor) are parallel.

Interaction: lines are not parallel.

Presence of significant interaction complicates the interpretation of results. Basically, an interaction means that "the levels of B affect the degree to which the levels of A affect the response variable." In other words, *both* factors affect the response variable, but the effects are not simply additive.

Modeling strategy: fit interaction model, and test for interaction. If present, use that model (after sorting out the means, performing diagnostics, etc.). If not significant, fit additive model and estimate/test main effects (& perform diagnostics).

AOV table for interaction model:

Source	SS	df	MS	F
A	SS(A)	$a-1$	$\frac{\text{SS(A)}}{a-1}$	$\frac{\text{MS(A)}}{\text{MS(error)}}$
B	SS(B)	$b-1$	$\frac{\text{SS(B)}}{b-1}$	$\frac{\text{MS(B)}}{\text{MS(error)}}$
A*B	SS(A*B)	$(a-1)(b-1)$	$\frac{\text{SS(A*B)}}{(a-1)(b-1)}$	$\frac{\text{MS(A*B)}}{\text{MS(error)}}$
error	SS(error)	$(n-1)ab$	$\frac{\text{SS(error)}}{(n-1)ab}$	
total	SS(total)	$nab-1$		

AOV table for additive model:

Source	SS	df	MS	F
A	SS(A)	$a-1$	$\frac{\text{SS(A)}}{a-1}$	$\frac{\text{MS(A)}}{\text{MS(error)}}$
B	SS(B)	$b-1$	$\frac{\text{SS(B)}}{b-1}$	$\frac{\text{MS(B)}}{\text{MS(error)}}$
error	SS(error)	$nab-a-b+1$	$\frac{\text{SS(error)}}{nab-a-b+1}$	
total	SS(total)	$nab-1$		