ON THE MODULARITY OF SOME FOUR DIMENSIONAL GALOIS REPRESENTATIONS

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1. The Theorem. The following theorem is an epic result in automorphic forms and number theory. We will describe a higher dimensional analogue of part of this result.

Theorem (Langlands–Tunnell). *Let $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Gl}(2, \mathbb{C})$ be an irreducible Galois representation. Then the projection of the image of ρ to $\mathrm{PGl}(2, \mathbb{C})$ is either dihedral, A_4 (tetrahedral), S_4 (octahedral) or A_5 (icosahedral). If ρ is not of icosahedral type, then there exists an irreducible cuspidal automorphic representation π of $\mathrm{Gl}(2, \mathbb{A})$ such that $L(s, \rho) = L(s, \pi)$.*

The proof of the Langlands-Tunnell theorem has two steps: (1) Passage to an automorphic representation π' of some group using one dimensional class field theory; (2) Passage from π' to the desired automorphic representation π.

We will describe a theorem analogous to the dihedral case of the Langlands-Tunnell theorem. Specifically, we consider the case of Galois representations

$$\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to {}^L\mathrm{GSp}(4) = \mathrm{GSp}(4, \mathbb{C}) \subset \mathrm{Gl}(4, \mathbb{C})$$

where the image of ρ is not contained in the Levi subgroup of any proper parabolic subgroup of $\mathrm{GSp}(4, \mathbb{C})$. Here, $\mathrm{GSp}(4, \mathbb{C})$ is the set of $g \in \mathrm{Gl}(4, \mathbb{C})$ such that

$${}^t g \begin{pmatrix} 0 & 1_2 \\ -1_2 & 0 \end{pmatrix} g = \lambda(g) \begin{pmatrix} 0 & 1_2 \\ -1_2 & 0 \end{pmatrix}$$

for some $\lambda(g) \in \mathbb{C}^\times$. We begin with the classification of such ρ. If ρ is reducible, we have the following lemma:

Lemma. *Let $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GSp}(4, \mathbb{C})$ be a Galois representation that does not factor through the Levi subgroup of any proper parabolic subgroup of $\mathrm{GSp}(4, \mathbb{C})$. If ρ is reducible as a complex representation, then there exist irreducible representations $\rho_1 : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Gl}(2, \mathbb{C})$ and $\rho_2 : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Gl}(2, \mathbb{C})$ such that $\det \rho_1 = \det \rho_2$, $\rho_1 \not\cong \rho_2$, $\rho_1 \oplus \rho_2$ carries a symplectic structure, and $\rho \cong \rho_1 \oplus \rho_2$.*

The following classification is well known.

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Proposition. *Let $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GSp}(4,\mathbb{C})$ be a Galois representation that does not factor through the Levi subgroup of any proper parabolic subgroup of $\mathrm{GSp}(4,\mathbb{C})$. Then exactly one of the following holds:*

(1) *ρ is reducible as a complex representation, and $\rho \cong \rho_1 \oplus \rho_2$ as in the lemma;*

(2) *ρ is irreducible, and there exists an irreducible representation $\rho_0 : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to \mathrm{Gl}(2,\mathbb{C})$ such that $\sigma\cdot\rho_0 \not\cong \rho_0$, $\det(\sigma\cdot\rho_0) = \det(\rho_0)$, $\mathrm{Ind}_{\mathrm{Gal}(\overline{\mathbb{Q}}/F)}^{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})} \rho_0$ carries a symplectic structure, and $\rho \cong \mathrm{Ind}_{\mathrm{Gal}(\overline{\mathbb{Q}}/F)}^{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})} \rho_0$. Here, $[F : \mathbb{Q}] = 2$ and σ is a representative for the nontrivial coset of $\mathrm{Gal}(\overline{\mathbb{Q}}/F)\backslash \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.*

(3) *ρ is irreducible and not induced.*

We are currently writing out the proof for the following theorem.

Theorem. *Let $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GSp}(4,\mathbb{C})$ be a Galois representation that does not factor through the Levi subgroup of any proper parabolic subgroup of $\mathrm{GSp}(4,\mathbb{C})$. Assume that two dimensional class field theory holds in all cases, i.e., the Langlands-Tunnell theorem holds in the icosahedral case. If ρ is of type (1) or (2), then ρ is modular: there exists an irreducible automorphic representation π of $\mathrm{GSp}(4,\mathbb{A})$ such that $L(s,\rho) = L(s,\pi)$. In fact, if $\pi \in \Pi(\rho)$ and*

$$m(\pi) = (1/|\mathbf{S}(\rho)|) \sum_{s\in\mathbf{S}(\rho)} \langle s,\pi\rangle \neq 0$$

then π occurs with multiplicity $m(\pi) = 1$ in the space of cusp forms on $\mathrm{GSp}(4,\mathbb{A})$ that have nonzero theta lifts to some $\mathrm{GO}(X(\mathbb{A}))$, where X is a four dimensional symmetric bilinear space defined over $\mathbb{Q}$.

Implicit in this statement is a definition of the local L-packets $\Pi(\rho_p)$ and the characters $\langle\cdot,\pi_p\rangle_p$ of $\mathbf{S}(\rho_p)$ for $\pi_p \in \Pi(\rho_p)$. Here, $\mathbf{S}(\rho)$ and $\mathbf{S}(\rho_p)$ are the connected component groups of ρ and ρ_p, respectively. See, for example, [LL].

In the remainder of these notes we will remark on the idea of the proof and make the theorem more explicit. For simplicity we consider type (1) representations. The case of type (2) representations is more interesting, but more complicated. In analogy to the Langlands-Tunnell theorem, the proof has two steps. Step 1 is easy: By the Langlands-Tunnell theorem and its assumption in the icosahedral case, there exist irreducible cuspidal automorpic representations τ_1 and τ_2 of $\mathrm{Gl}(2,\mathbb{A})$ such that $\tau_1 \not\cong \tau_2$, τ_1 and τ_2 have the same central character, and $L(s,\rho) = L(s,\tau_1)L(s,\tau_2)$.

2. The Idea. To do step 2 we use the theta correspondence. The first use of the theta correspondence in this context seems to be due to H. Yoshida [Y]. The theta correspondence lifts automorphic forms from orthogonal groups to symplectic groups, and vice versa. However, automorphic forms can lift to zero, and it is a fundamental problem to determine when this happens. This is a key issue for our problem. Our pair τ_1 and τ_2 is related to an orthogonal group in the following way. Let $X(\mathbb{A})$ be the space of two by two matrices

$M_2(\mathbb{A})$, and equip $X(\mathbb{A})$ with the quadratic form which is the determinant. Then there is an exact sequence

$$1 \to \mathbb{A}^\times \to \mathrm{Gl}(2,\mathbb{A}) \times \mathrm{Gl}(2,\mathbb{A}) \to \mathrm{GSO}(X(\mathbb{A})) \to 1.$$

where the first map takes t to (t, t^{-1}) and the second map takes (g_1, g_2) to the linear transformation defined by $(g_1, g_2)x = g_1 x g_2^*$, where $*$ is the canonical involution on $M_2(\mathbb{A})$. Since τ_1 and τ_2 have the same central character, τ_1 and τ_2 give an irreducible cuspidal automorphic representation τ of $\mathrm{GSO}(X(\mathbb{A}))$. We can induce and obtain various irreducible cuspidal automorphic representations of $\mathrm{GO}(X(\mathbb{A}))$. In the type (1) case, only one is relevant: $\tau^+ = \otimes_p \tau_p^+$. The representation τ_p^+ is defined via the theory of distinguished representations [R2]; its restriction to $\mathrm{GSO}(X(\mathbb{Q}_p))$ contains τ_p.

However, there are more relevant automorphic representations. There is a generalization of the above exact sequence. Suppose that $X_D(\mathbb{A})$ are the adelic points $D(\mathbb{A})$ of a quaternion algebra D defined over $\mathbb{Q}$, and equip $X_D(\mathbb{A})$ with the quadratic form which is the norm form. Then we have an exact sequence

$$1 \to \mathbb{A}^\times \to D(\mathbb{A})^\times \times D(\mathbb{A})^\times \to \mathrm{GSO}(X_D(\mathbb{A})) \to 1$$

with the same definitions. Via the Jacquet-Langlands correspondence, we can lift τ_1 and τ_2 to irreducible cuspidal automorphic representations τ_1^D and τ_2^D of $D(\mathbb{A})^\times$, and obtain another relevant irreducible cuspidal automorphic representation of $\mathrm{GO}(X_D(\mathbb{A}))$: $\tau^{D,+} = \otimes_p \tau_p^{D,+}$. Of course, $\tau^{D,+} \neq 0$ if and only if $\rho_{1,p}$ and $\rho_{2,p}$ are irreducible, i.e., $\tau_{1,p}$ and $\tau_{2,p}$ are in the discrete series for all the primes p at which D is ramified.

The main point of the proof of the theorem is to show that the theta lifts $\theta(\tau^{D,+})$ of the nonzero $\tau^{D,+}$ are nonzero. These irreducible automorphic representations of $\mathrm{GSp}(4,\mathbb{A})$ are exactly the elements of the L-packet $\Pi(\rho)$ which are predicted to be automorphic by the multiplicity formula. To show nonvanishing we apply the general theory described in [R3]. This theory is a representation theoretic interpretation of work of S. Boecherer and R. Schulze-Pillot [B-SP] who solved the nonvanishing problem posed by H. Yoshida in the context of classical modular forms. Some key ingredients in this theory are: an understanding of local theta lifts of tempered representations [M], [R4]; an understanding of certain local zeta integrals and their poles [R5]; and the understanding of poles of standard L-functions of cuspidal automorphic representations of $\mathrm{Sp}(n,\mathbb{A})$ due to S.S. Kudla and S. Rallis [KR].

3. Local L-packets. In the remainder of these notes we will make the main theorem more explicit by describing the local L-packets and the pairing in the multiplicity formula. The definition of the local L-packets is closely related to the local theory of theta lifts described above. The first use of the theta correspondence to define L-packets for $\mathrm{GSp}(4)$ seems to be [V]. Suppose for the moment that X is any even dimensional nondegenerate symmetric bilinear space over $\mathbb{Q}_p$ with $\mathrm{disc}(X) = 1$. Then for each nonnegative integer n there exists a representation Ω_p of $\mathrm{GO}(X(\mathbb{Q}_p)) \times \mathrm{GSp}(2n,\mathbb{Q}_p)$ called the Weil representation, and we have the following theorem which follows from the corresponding result for isometries.

Theorem [R1]**.** *The condition*

$$\mathrm{Hom}_{\mathrm{GO}(X(\mathbb{Q}_p))\times \mathrm{GSp}(2n,\mathbb{Q}_p)}(\Omega_p, \sigma\otimes\pi) \neq 0$$

defines a graph of a bijection between the $\sigma \in \mathrm{Irr}(\mathrm{GO}(X(\mathbb{Q}_p)))$ and $\pi \in \mathrm{Irr}(\mathrm{GSp}(2n,\mathbb{Q}_p))$ that are nonzero quotients of Ω_p. Moreover, for all σ and π,

$$\dim \mathrm{Hom}_{\mathrm{GO}(X(\mathbb{Q}_p))\times \mathrm{GSp}(2n,\mathbb{Q}_p)}(\Omega_p, \sigma\otimes\pi) \leq 1.$$

If $\sigma \in \mathrm{Irr}(\mathrm{GO}(X(\mathbb{Q}_p)))$ occurs in the correspondence with $\mathrm{GSp}(2n,\mathbb{Q}_p)$ we write $\theta(\sigma)$ for the corresponding representation of $\mathrm{GSp}(2n,\mathbb{Q}_p)$. The case $\mathrm{disc}(X)\neq 1$ can also be dealt with, but is more complicated.

Lemma [R2]**.** *We have the following results concerning the local theta correspondence and τ_p from section 2:*

(1) *If $\rho_{1,p}$ or $\rho_{2,p}$ is reducible and $\rho_{1,p}\not\cong\rho_{2,p}$, then*

$$\tau_p \longrightarrow \tau_p^+ = \mathrm{Ind}_{\mathrm{GSO}(X(\mathbb{Q}_p))}^{\mathrm{GO}(X(\mathbb{Q}_p))}\tau_p \longrightarrow \theta(\tau_p^+).$$

(2) *If $\rho_{1,p}$ or $\rho_{2,p}$ is reducible and $\rho_{1,p}\cong\rho_{2,p}$, then*

$$\begin{array}{ccc} & \tau_p^+ \text{ constituent of } \mathrm{Ind}_{\mathrm{GSO}(X(\mathbb{Q}_p))}^{\mathrm{GO}(X(\mathbb{Q}_p))}\tau_p \quad \longrightarrow \quad \theta(\tau_p^+) \\ \nearrow & \\ \tau_p & \\ \searrow & \\ & \tau_p^- \text{ constituent of } \mathrm{Ind}_{\mathrm{GSO}(X(\mathbb{Q}_p))}^{\mathrm{GO}(X(\mathbb{Q}_p))}\tau_p \quad \text{does not lift} \end{array}$$

(3) *If $\rho_{1,p}$ and $\rho_{2,p}$ are irreducible, then we have the same results as in (1) and (2); in addition, the same results hold for $\tau_{1,p}^D$ and $\tau_{2,p}^D$, for D the unique ramified quaternion algebra at p.*

The next lemma tells us how many elements should be in the L-packet $\Pi(\rho_p)$.

Lemma. *The connected component group $\mathbf{S}(\rho_p)$ is given as follows:*

$$\mathbf{S}(\rho_p) = \begin{cases} 1 \text{ if } \rho_{1,p} \text{ or } \rho_{2,p} \text{ is reducible,} \\ Z_2 \text{ if } \rho_{1,p} \text{ and } \rho_{2,p} \text{ are irreducible.} \end{cases}$$

We can now define the L-packet corresponding to ρ_p:

Definition. *We set:*

(1) *If $\rho_{1,p}$ or $\rho_{2,p}$ is reducible, define $\Pi(\rho_p) = \{\theta(\tau_p^+)\}$.*
(2) *If $\rho_{1,p}$ and $\rho_{2,p}$ are irreducible, define $\Pi(\rho_p) = \{\theta(\tau_p^+), \theta(\tau_p^{D,+})\}$.*

Finally, we define the characters $\langle\,\cdot\,,\pi_p\rangle$ of $\mathbf{S}(\rho_p)$ required for the multiplicity formula.

Definition. *Let* $\pi \in \Pi(\rho_p)$*.*

(1) *Assume* $\rho_{1,p}$ *or* $\rho_{2,p}$ *is reducible. Then* $|\mathbf{S}(\rho_p)| = |\Pi(\rho_p)| = 1$*, and we define* $\langle \cdot \, , \pi_p \rangle$ *to be the trivial character;*

(2) *Assume* $\rho_{1,p}$ *and* $\rho_{2,p}$ *are irreducible. Then* $|\mathbf{S}(\rho_p)| = |\Pi(\rho_p)| = 2$*, and we define*

$$\langle \cdot \, , \pi_p \rangle = \begin{cases} 1 \text{ if } \pi = \theta(\tau_p^{+}), \\ \chi \text{ if } \pi = \theta(\tau_p^{D,+}). \end{cases}$$

Here, χ *is the nontrivial character of* $\mathbf{S}(\rho_p)$*.*

References

[B-PS] S. Böcherer and R. Schulze-Pillot, *Siegel modular forms and theta series attached to quaternion algebras*, Nagoya Math. J. **121** (1991), 35–96.

[KR] S.S. Kudla and S. Rallis, *A regularized Siegel-Weil formula: The first term identity*, Annals of Math. **140** (1994), 1–80.

[LL] J.–B. Labesse and R. P. Langlands, *L-indistinguishability for* $SL(2)$, Can. J. Math. **31** (1979), 726–785.

[M] C. Moeglin, *Correspondance de Howe pour les paires reductives duales: Quelques calculs dans le cas archimédien*, J. Functional Analysis **85**, 1–85.

[R1] B. Roberts, *The theta correspondence for similitudes*, Israel J. of Math. **94** (1996), 285–317.

[R2] B. Roberts, *The nonarchimedean theta correspondence for* GSp(2) *and* GO(4), Trans. Amer. Math. Soc. **351** (1999), 781–811.

[R3] B. Roberts, *Nonvanishing of global theta lifts*, Proceedings of the AMS Summer Research Conference, Representation Theory of Real and p-adic Reductive Groups, July 1997, www.math.umd.edu ˜jda/ seattle_proceedings, 1–6.

[R4] B. Roberts, *Tempered representations and the theta correspondence*, Canadian J. Math. **50** (1998), 1105–1118.

[R5] B. Roberts, *Poles of local L–functions and the theta correspondence*, preprint.

[V] Vignéras, M.F., *Correspondances entre representations automorphes de* Gl(2) *sur une estension quadratic de* GSp(4) *sur* $\mathbb{Q}$*, conjecture locale de Langlands pour* GSp(4), in The Selberg Trace Formula and Related Topics, American Mathematical Society, Providence, R.I., 1986.

[Y] H. Yoshida, *On Siegel modular forms obtained from theta series*, J. Reine Angew. Math. **352** (1984), 184–219.

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