POLES OF LOCAL L-FUNCTIONS AND THE THETA CORRESPONDENCE

Brooks Roberts

The University of Toronto

Abstract. Let V be a nondegenerate even dimensional symmetric bilinear space over a nonarchimedean local field F of characteristic zero. Let $\sigma \in \mathrm{Irr}(\mathrm{O}(V))$ be pre-unitary, and assume that σ corresponds to a tempered element of $\mathrm{Irr}(\mathrm{Sp}(n_0, F))$ with respect to the theta correspondence for some n_0 with $2n_0 \geq \dim V$. We show that if $2n > 2n_0$, and $\pi \in \mathrm{Irr}(\mathrm{Sp}(n, F))$ corresponds to σ, then the doubling L-function of π twisted by the quadratic character χ_V of $F^\times$ associated to V has $L(s, |\cdot|^{-(n-\dim V/2)})$ as a factor, and so has a pole at $n - \dim V/2$. The existence of this pole has an application to the important nonvanishing problem for global theta lifts.

Let F be a nonarchimedean local field of characteristic zero, let q be the order of the residue class field of F, let χ be a quasi-character of $F^\times$, and let n be a positive integer. Let $\pi \in \mathrm{Irr}(\mathrm{Sp}(n, F))$. Just as for representations of the general linear group [GJ], [J] zeta integrals have been defined for π [PSR1,2,3]. Associated to each coefficient f of π and a good χ-section Φ [PSR3], there is a zeta integral $Z(s, f, \Phi)$ which is a rational function in q^{-s}, and the $Z(s, f, \Phi)$ generate a $\mathbb{C}[q^s, q^{-s}]$ fractional ideal in $\mathbb{C}(q^{-s})$. This ideal has a unique generator of the form $1/P(q^{-(s+1/2)})$ for some $P \in \mathbb{C}[X]$ with $P(0) = 1$, and assuming the local Langlands conjecture for $\mathrm{Sp}(n)$ so that π has an L-parameter, one can investigate if $1/P(q^{-s})$ is the standard L-function of π, twisted by χ. This is known if $n = 1$ and π is an irreducible principal series representation or a nonexceptional supercuspidal representation [H].

These zeta integrals have global applications, and the main result of this paper provides some of the necessary background for one such employment. In [BSP], S. Böcherer and R. Schulze-Pillot made innovative use of the standard L-functions attached to Siegel modular forms to solve the longstanding Yoshida theta lifting problem. We have given a representation theoretic interpretation of their method [R2] which applies to global theta lifts of tempered irreducible cuspidal automorphic representations of even orthogonal groups $\mathrm{O}(V)$ to automorphic representations of $\mathrm{Sp}(n)$ in the range $2n \geq \dim V$. The main result of this paper is a key component of this interpretation.

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To explain the result we need some more notation. Let V be a nondegenerate symmetric bilinear space defined over F of even dimension l. Fix a nontrivial additive character ψ of F, and let ω_n be the Weil representation of $\mathrm{O}(V) \times \mathrm{Sp}(n,F)$ associated to ψ. Let χ_V be the quadratic character of $F^\times$ defined by $\chi_V(x) = (x, \mathrm{disc}(V))_F$, where $(\cdot,\cdot)_F$ is the Hilbert symbol of F. Let $\sigma \in \mathrm{Irr}(\mathrm{O}(V))$ and $\pi \in \mathrm{Irr}(\mathrm{Sp}(n,F))$. We say that σ and π correspond if $\mathrm{Hom}_{\mathrm{O}(V)\times\mathrm{Sp}(n,F)}(\omega_n, \sigma\otimes\pi) \neq 0$.

Theorem 8.2. *Let $\sigma \in \mathrm{Irr}(\mathrm{O}(V))$ be pre-unitary. Let n_0 and n be positive integers such that $2n > 2n_0 \geq l = \dim_F V$, and let $\pi_0 \in \mathrm{Irr}(\mathrm{Sp}(n_0,F))$ and $\pi \in \mathrm{Irr}(\mathrm{Sp}(n,F))$ correspond to σ. Assume π_0 is tempered. Then there exist a coefficient f of π, a standard χ_V-section Φ, and nonzero complex numbers A and B such that*

$$Z(s,f,\Phi) = Aq^{Bs} L(s+1/2, |\cdot|^{-(n-l/2)}).$$

In particular, $Z(s,f,\Phi)$ has a pole at $s = n - l/2 - 1/2$.

S.S. Kudla and S. Rallis have announced results about the poles of L-functions of irreducible supercuspidal representations of $\mathrm{Sp}(n,F)$, and more generally, representations of $\mathrm{Sp}(n,F)$ that do not occur in the boundary of a certain degenerate principal series representation. See [HKS], p. 945. We do not know if Theorem 8.2 is implied by their results.

To prove Theorem 8.2, we use two results. In [R1] we showed that if the notation is as in Theorem 8.2, then π is determined by π_0 in the sense that π has the Langlands quotient form $\pi = L(\chi_V|\cdot|^{n-l/2} \otimes \chi_V|\cdot|^{n-1-l/2} \otimes \cdots \otimes \chi_V|\cdot|^{n_0+1-l/2} \otimes \pi_0)$. See section 7 for the notation. As in the case of the general linear group, it is natural to try to relate the zeta integrals of a Langlands quotient to the analogous doubling-type zeta integrals of the quotient data. The expected (untranslated) doubling L-function twisted by χ_V of the leading factor $\chi_V|\cdot|^{n-l/2}$ is $L(s+1/2, |\cdot|^{n-l/2}) L(s+1/2, |\cdot|^{-(n-l/2)})$. Thus, in particular it should be possible to realize $L(s+1/2, |\cdot|^{-(n-l/2)})$ as a zeta integral of π. The following result, which is the main technical result of the paper, achieves this realization.

Theorem 8.1. *Let χ be a quasi-character of $F^\times$. Let $\pi \in \mathrm{Irr}(\mathrm{Sp}(n,F))$. Write π as a Langlands quotient $\pi = L(\delta_1 \otimes \cdots \otimes \delta_t \otimes \tau)$, where $n = n_1 + \cdots + n_t + n_0$ is an ordered partition of n with $n_1, \ldots, n_t$ positive if $t > 0$, $\delta_i \in \mathrm{Irr}(\mathrm{Gl}(n_i,F))$ are essentially tempered for $1 \leq i \leq t$, and $\tau \in \mathrm{Irr}(\mathrm{Sp}(n_0,F))$ is tempered. Assume that $n_1 = 1$, so that δ_1 is a quasi-character. Then there exist a coefficient f of π, a standard χ-section Φ, and nonzero constants A and B such that*

$$Z(s,f,\Phi) = A\Big(\int_{\mathfrak{O}^\times} \delta_1\chi^{-1}(u)\,du\Big) q^{Bs} L(s+1/2, \chi\delta_1^{-1}).$$

Our proof of Theorem 8.1 is modeled on the arguments for the general linear group from section 3 of [J]. In general, a stronger result, like Theorem 3.4 of [J], might be expected. One obstacle is the manipulation of the test functions, which in contrast to the case of the general linear group, are not just Schwartz functions, but rather sections of certain degenerate principal series representations. The first step towards proving an analogue of Theorem 3.4 of [J] would be to show that the zeta integral ideal of a parabolically induced representation is the product of the zeta integral ideals of the inducing data. See the comment before Lemma 3.3. The second step would also require a result about sections, namely a generalization of Corollary 5.3. Granted this, our argument for Theorem 8.2 is general, and this would be a component in the proof of the general result. More probably will be required. The case of the general linear group uses information about the poles of L-functions of tempered representations. At the moment, the tempered dual of the symplectic group remains unclassified.

Notation. We use the following notation. Throughout, F is a nonarchimedean local field of characteristic zero with integers $\mathfrak{O}$, prime ideal $\mathfrak{P} = \pi_F\mathfrak{O} \subset \mathfrak{O}$, Hilbert symbol $(\cdot,\cdot)_F$, and valuation $|\cdot|$ such that if μ is an additive Haar measure on F, then $\mu(xA) = |x|\mu(A)$ for $x \in F$ and $A \subset F$. Let $q = |\mathfrak{O}/\mathfrak{P}|$. For n a positive integer, $\mathrm{Sp}(n,F)$ is the symplectic group of rank n. Let G be a group of td-type [C] with a countable basis. Let $\mathcal{S}(G)$ be the $\mathbb{C}$ vector space of locally constant, compactly supported $\mathbb{C}$ valued functions on G. Let $\mathrm{Irr}(G)$ be the set of equivalence classes of smooth admissible irreducible representations of G. Let π be a smooth representation of G. The smooth contragredient representation of π is $\pi^\vee$, and if π admits a central character, we denote it by ω_π. A coefficient of π is a finite $\mathbb{C}$ linear combination of matrix coefficients of π. A representation π of G is pre-unitary if there is a nondegenerate G invariant Hermitian form on the space of π. Suppose G is unimodular, and M and N are closed subgroups of G such that M normalizes N, $M \cap N = 1$, $P = MN$ is closed in G, N is unimodular and $P\backslash G$ is compact. Fix a Haar measure dn on N, and for $m \in M$, let $\delta(m)$ be the positive number such that all $f \in \mathcal{S}(G)$,

$$\int_N f(m^{-1}nm)\,dn = \delta(m)\int_N f(n)\,dn.$$

If σ is a smooth representation of M, then $\mathrm{Ind}_P^G\,\sigma$ is the representation of G by right translation on the $\mathbb{C}$ vector space of functions f on G with values in σ that are right invariant under a compact open subgroup of G and such that $f(mng) = \delta(m)^{1/2}\sigma(m)f(g)$ for $m \in M$, $n \in N$ and $g \in G$. If $\pi \in \mathrm{Irr}(\mathrm{Gl}(q,F))$, then $e(\pi)$ is the unique real number such that the central character of $\pi \otimes |\det|^{-e(\pi)}$ is unitary. If n is a positive integer, then an ordered partition of n is a k-tuple $(n_1,\ldots,n_k)$ of positive integers such that $n = n_1 + \cdots + n_k$. If G is the group of F-points of a connected reductive algebraic group defined over F then $\pi \in \mathrm{Irr}(G)$ is tempered if ω_π is unitary and every matrix coefficient of π lies in $\mathrm{L}^{2+\epsilon}(G/\mathrm{Z}(G))$ for all $\epsilon > 0$. If $\pi \in \mathrm{Irr}(\mathrm{Gl}(n,F))$, then π is essentially tempered or essentially square integrable if $\pi \otimes |\det|^{-e(\pi)}$ is tempered or square integrable, respectively. If χ is a quasi-character of $F^\times$, then $c(\chi)$ is the conductor of χ, i.e., $c(\chi) = 0$ if χ is unramified and otherwise $c(\chi)$ is the smallest positive integer such that $\chi(1+\mathfrak{P}^{c(\chi)}) = 1$. In this paper, all functions act on the left, and composition of functions is taken from right to left. In this paper we *do not* make assumptions about the residual characteristic of F or assume Howe duality.

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1. Zeta integrals for the symplectic group. We follow [PSR1,2,3]. The following notation will be fixed for the remainder of the paper. Let $(V, \langle\cdot,\cdot\rangle)$ be a nondegenerate symplectic bilinear space of dimension $2n$ over F. Let $G = \mathrm{Sp}(V)$. Define a nondegenerate symplectic bilinear space $(V', \langle\cdot,\cdot\rangle)$ over F by letting $V' = V \times V$ and $\langle(v_1,v_2),(v_1',v_2')\rangle = \langle v_1, v_1'\rangle - \langle v_2, v_2'\rangle$. Let $H = \mathrm{Sp}(V')$. We have an embedding

$$i : G \times G \hookrightarrow H$$

defined by $i(g,g')(v,v') = (gv, g'v')$. Let V^d be the subspace of pairs (v,v) for $v \in V$. Then V^d is a Lagrangian of V'. Let P' be the parabolic subgroup of H stabilizing V^d. Note that G embedded on the diagonal is contained in P'. It will useful to explicitly express the elements of G and H as matrices. Fix a symplectic basis $\mathcal{B} = \{x_1,\dots,x_n,y_1,\dots,y_n\}$ of V. As a symplectic basis for V' we use $\mathcal{B}' = \{e_1,\dots,e_n,f_1,\dots,f_n,e_1',\dots e_n',f_1',\dots,f_n'\}$, where $e_i = (x_i,x_i)$, $f_i = (y_i,y_i)$, $e_i' = (y_i,0)$ and $f_i' = (0,x_i)$ for $1 \le i \le n$. Fix maximal compact subgroups $K = \mathrm{Sp}(n,\mathfrak{O})$ and $K' = \mathrm{Sp}(2n,\mathfrak{O})$ of G and H, respectively. The elements of P' have the usual form of the elements of the Siegel parabolic of $\mathrm{Sp}(2n,F)$. The set $P'i(G\times 1)$ is dense in H. We have

$$i(\begin{bmatrix} a & b \\ c & d\end{bmatrix},1) = \begin{bmatrix} a & b & b & 0 \\ 0 & 1 & 0 & 0 \\ c & d-1 & d & 0 \\ 1-a & -b & -b & 1\end{bmatrix}, \quad i(1,\begin{bmatrix} a & b \\ c & d\end{bmatrix}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ c & d & 0 & c \\ -c & 1-d & 1 & -c \\ a-1 & b & 0 & a\end{bmatrix}.$$

Next, we define the test functions for the zeta integrals. Fix a quasi-character χ of $F^\times$. For $s \in \mathbb{C}$, let $\mathrm{I}_{2n}(s,\chi)$ be the H space of functions $\Phi : H \to \mathbb{C}$ that are right invariant under a compact open subgroup of H and satisfy $\Phi(pg) = \alpha_\chi(p,s)\Phi(g)$ for $g \in H$ and $p \in P'$; here,

$$p = \begin{bmatrix} a & b \\ 0 & {}^t a^{-1}\end{bmatrix}, \quad \alpha_\chi(p,s) = \chi(\det a)|\det a|^{s+\rho_{2n}},$$

where $\rho_{2n} = (2n+1)/2$. A χ-**section** is a function $\Phi : H \times \mathbb{C} \to \mathbb{C}$ such that for all $s \in \mathbb{C}$ the function $\Phi(*,s)$ is contained in $\mathrm{I}_{2n}(s,\chi)$. A χ-section Φ is **standard** if the restriction of Φ to K is independent of s, i.e., $\Phi(k,s) = \Phi(k,s')$ for $k \in K'$ and $s,s' \in \mathbb{C}$. Let $\mathrm{I}_{2n}^{\mathrm{Stan}}(\chi)$ be the $\mathbb{C}$ vector space of all standard χ-sections. Clearly, $\mathrm{I}_{2n}^{\mathrm{Stan}}(\chi)$ is closed under the right translation action of K', and restriction to K' defines a K' isomorphism from $\mathrm{I}_{2n}^{\mathrm{Stan}}(\chi)$ to $\mathrm{Ind}_{P'\cap K'}^{K'}\chi$. In addition, there is the concept of a good χ-section. See [PSR3], p. 110, and [HKS], p. 970. Good sections consist of the standard sections, the image of the standard sections under a certain operator, and, if χ is unramified, translates of a normalization of the unramified section. Since we shall only work with standard sections we omit the precise definition.

We can now define the zeta integrals. Fix a Haar measure on G. Let $\pi \in \mathrm{Irr}(G)$. It is known that there exists a real number σ_0 such that for all $\Phi \in \mathrm{I}_{2n}^{\mathrm{Stan}}(\chi)$, coefficients f of π, and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > \sigma_0$,

$$Z(s,f,\Phi) = \int_G \Phi(i(g,1),s) f(g)\, dg$$

converges absolutely. Let $\Phi \in \mathrm{I}_{2n}^{\mathrm{Stan}}(\chi)$ and let f be a coefficient of π. It is known that there is a rational function $p(X) \in \mathbb{C}(X)$ such that $Z(s,f,\Phi) = p(q^{-s})$ for s with $\mathrm{Re}(s) > \sigma_0$, so that in particular $Z(s,f,\Phi)$ can be regarded as a meromorphic function on $\mathbb{C}$ with a finite number of poles. See Proposition of 1.1 of [PSR1] for these results, and the remark before Lemma 3.3 below for some more information.

2. Zeta integrals for the general linear group. There is a similar development for the general linear group. Let X be a vector space over F of dimension k. In the next section, X will be a subspace of V, and the groups we define will be subgroups of H. Let $G_1 = \mathrm{Gl}(X)$, $X' = X \times X$, and $H_1 = \mathrm{Gl}(X')$. We again have an embedding

$$i_1 : G_1 \times G_1 \hookrightarrow H_1$$

given by $i_1(g,g')(x,x') = (gx, g'x')$. Let X^d be the subspace of pairs (x,x) for $x \in X$. Let P_1' be the maximal parabolic subgroup of G_1 stabilizing X^d; again, G_1 embedded on the diagonal is contained in P_1'. Let $\mathcal{B}_1 = \{x_1,\dots,x_k\}$ be an ordered basis for X. As an ordered basis for X' we will use $\mathcal{B}_1' = \{e_1,\dots,e_k,f_1',\dots,f_k'\}$, where $e_i = (x_i,x_i)$ and $f_i' = (0,x_i)$ for $1 \le i \le k$. Fix maximal compact subgroups $K_1 = \mathrm{Gl}(k,\mathfrak{O})$ and $K_1' = \mathrm{Gl}(2k,\mathfrak{O})$ of G_1 and H_1, respectively. The elements of P_1' have the usual form of the $k+k$ standard parabolic of $\mathrm{Gl}(2k,F)$. We have

$$i_1(h,1) = \begin{bmatrix} h & 0 \\ 1-h & 1 \end{bmatrix}, \quad i_1(1,h) = \begin{bmatrix} 1 & 0 \\ h-1 & h \end{bmatrix}.$$

Fix quasi-characters $\mu_1,\mu_2 : F^\times \to \mathbb{C}^\times$. For $s \in \mathbb{C}$, let $\mathrm{I}_{\mathrm{Gl}(2k)}(s,\mu_1,\mu_2)$ be the H_1 space of functions $\Phi_1 : H_1 \to \mathbb{C}$ that are right invariant under a compact open subgroup of H_1 and satisfy $\Phi_1(ph) = \alpha_{\mu_1,\mu_2}(p,s)\Phi_1(h)$ for $h \in H_1$ and $p \in P_1'$, where

$$p = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}, \quad \alpha_{\mu_1,\mu_2}(p,s) = |\det a/\det c|^{s+k/2}\mu_1(\det a)\mu_2(\det c).$$

Again, we have the concept of a μ_1,μ_2 section and μ_1,μ_2 standard section. The zeta integrals associated to an element of $\mathrm{Irr}(G_1)$ are defined analogously and the same results hold.

3. An auxiliary parabolic. To prove the main theorem we will need to relate zeta integrals for $\pi \in \mathrm{Irr}(G)$ to zeta integrals associated to the Langlands quotient data for π, which consists of representations of general linear groups and another, smaller, symplectic group. To do so we need to introduce some more groups.

In fact, it will suffice to work with representations induced off maximal parabolic subgroups of G. Fix an integer k such that $1 \le k \le n$. Let X be the subspace of V with ordered basis $\{x_1, \dots, x_k\}$, let Y be the subspace with ordered basis $\{y_1, \dots, y_k\}$, and let Z be the subspace with ordered basis $\{x_{k+1}, \dots, x_n, y_{k+1}, \dots, y_n\}$. As in section 2, let $G_1 = \mathrm{Gl}(X)$. Also, let $G_2 = \mathrm{Sp}(Z)$. The definitions of sections 1 and 2 apply to G_1 and G_2. Let P be the maximal parabolic subgroup of G stabilizing X. The elements of the Levi component M and unipotent radical U of P have the form

$$m(g_1, g_2) = \begin{bmatrix} g_1 & 0 & 0 & 0 \\ 0 & a & 0 & b \\ 0 & 0 & {}^t g_1^{-1} & 0 \\ 0 & c & 0 & d \end{bmatrix}, \quad u(x,y,z) = \begin{bmatrix} 1 & x & y & z \\ 0 & 1 & {}^t z & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -{}^t x & 1 \end{bmatrix},$$

respectively. Here, $g_1 \in G_1$, $g_2 \in G_2$ has form

$$g_2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

$x \in \mathrm{M}_{k\times(n-k)}(F)$, $y \in \mathrm{M}_{k\times k}(F)$ and $z \in \mathrm{M}_{k\times(n-k)}(F)$ with ${}^t(y - x{}^t z) = y - x{}^t z$. We have $U = U_1 \rtimes U_2$, with U_1 the subgroup of elements of the form $u(0,y,z)$ and U_2 the subgroup of elements of the form $u(x,0,0)$. Clearly, $U_1 \cong \mathrm{Sym}_{k\times k}(F) \times \mathrm{M}_{k\times(n-k)}(F)$ and $U_2 \cong \mathrm{M}_{k\times(n-k)}(F)$. The groups U_1, U_2 and U are all unimodular. Using the formula for Haar measures on semi-direct products, if Haar measures on $\mathrm{Sym}_{k\times k}(F)$ and $\mathrm{M}_{k\times(n-k)}(F)$ are fixed, then a Haar measure on U is given by

$$\int_U F(u)\,du = \int_A F(u(x, y - z{}^t x, z))\,d(xyz)$$

for $F \in \mathcal{S}(U)$, where $A = \mathrm{M}_{k\times(n-k)}(F) \times \mathrm{Sym}_{k\times k}(F) \times \mathrm{M}_{k\times(n-k)}(F)$, and we use the product measure on A. We will also need to use the parabolic subgroup $\overline{P}$ opposite to P. In matrices, this is the group of ${}^t p$ with $p \in P$. The Levi component of $\overline{P}$ is of course M, and we let $\overline{U} = {}^t U$ denote the unipotent radical of $\overline{P}$. Fixing Haar measures on $\mathrm{M}_{k\times(n-k)}(F)$ and $\mathrm{Sym}_{k\times k}(F)$ determines a Haar measure on $\overline{U}$.

We will need to consider integrals of functions on H over $i(U \times 1)$. For sections, we can write such integrals as integrals over a more tractable subgroup of H. Let V be the abelian subgroup of H consisting of elements of the form

$$v(x,y,z) = \begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}, \quad c = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & {}^t x & 0 \\ 0 & x & y & z \\ 0 & 0 & {}^t z & 0 \end{bmatrix},$$

for $x \in \mathrm{M}_{k\times(n-k)}(F)$, $y \in \mathrm{Sym}_{k\times k}(F)$ and $z \in \mathrm{M}_{k\times(n-k)}(F)$. Thus, $V \cong A$. By the following identity, V is contained in $P'i(U \times 1)$:

$$v(x,y,z) = \begin{bmatrix} a & b \\ 0 & {}^t a^{-1} \end{bmatrix} i(u(-x,-y-z^t x,-z),1),$$

where

$$a = \begin{bmatrix} 1 & x & y+x^t z & z \\ 0 & 1 & {}^t z & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} y+x^t z & z & 0 & 0 \\ {}^t z & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Lemma 3.1. *Let S be a subset of H such that $i(U \times 1) \subset S$ and $pS \subset S$ for $p \in P'$, so that S also contains V. Let $F : S \to [0,\infty]$ be a function such that $F(ph) = F(h)$ for $h \in S$ and $p \in P'$ as in section 1 with $\det a = 1$, and $F \circ i|_{U\times 1}$ is measurable. Fix Haar measures on $\mathrm{Sym}_{k\times k}(F)$ and $\mathrm{M}_{k\times(n-k)}(F)$, so that Haar measures on U and V are determined. Then $F|_V$ is measurable, and*

$$\int_U F(i(u,1))\,du = \int_V F(v)\,dv.$$

Proof. This follows from the formula for the Haar measure on U and the identity preceding the lemma. □

A similar argument proves the following: Let S and the Haar measures be as in Lemma 3.1. Let $F : S \to \mathbb{C}$ be a function such that $F(ph) = F(h)$ for $h \in S$ and $p \in P'$ with $\det a = 1$, and $F \circ i|_{U\times 1} \in L^1(U)$. Then $F|_V \in L^1(V)$, and

$$\int_U F(i(u,1))\,du = \int_V F(v)\,dv.$$

Next, we introduce an auxiliary parabolic of H that will be the key to relating zeta integrals on G to zeta integrals on G_1 and G_2. In doing so, we were inspired by the comments in section 1 of [PSR1]. Let $P_{1,2}$ be the maximal parabolic of H stabilizing the totally isotropic subspace $X' = X \times X$ of V'. We have the decomposition $V' = X' \oplus Z' \oplus Y'$, where $X' = X \times X$, $Y' = Y \times Y$ and $Z' = Z \times Z$. Evidently, X' and Y' are totally isotropic and dually paired, and the orthogonal complement of $X' \oplus Y'$ is Z'. The Levi component $M_{1,2}$ of $P_{1,2}$ is isomorphic to $\mathrm{Gl}(X') \times \mathrm{Sp}(Z')$. Letting $H_1 = \mathrm{Gl}(X')$ and $H_2 = \mathrm{Sp}(Z')$, we obtain embeddings

$$i'_1 : H_1 \hookrightarrow M_{1,2} \subset P_{1,2} \subset H, \quad i'_2 : H_2 \hookrightarrow M_{1,2} \subset P_{1,2} \subset H.$$

Note that by sections 1 and 2 we also have embeddings

$$i_1 : G_1 \times G_1 \hookrightarrow H_1, \quad i_2 : G_2 \times G_2 \hookrightarrow H_2.$$

We use $\{e_1, \dots, e_k, f_1', \dots, f_k'\}$ and $\{e_{k+1}, \dots, e_n, f_{k+1}, \dots, f_n, e_{k+1}', \dots, e_n', f_{k+1}', \dots, f_n'\}$ as ordered bases for X' and Z', respectively. These choices of bases are consistent with the definitions of section 1 and 2 applied to G_1 and G_2. Explicitly, suppose $h_1 \in H_1$ and $h_2 \in H_2$ with

$$h_1 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad {}^t h_1^{-1} = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}, \quad h_2 = \begin{bmatrix} a_1 & a_2 & b_1 & b_2 \\ a_3 & a_4 & b_3 & b_4 \\ c_1 & c_2 & d_1 & d_2 \\ c_3 & c_4 & d_3 & d_4 \end{bmatrix}.$$

Then

$$i_1'(h_1) = \begin{bmatrix} a & 0 & 0 & 0 & 0 & 0 & b & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & d' & 0 & -c' & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -b' & 0 & a' & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ c & 0 & 0 & 0 & 0 & 0 & d & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \ i_2'(h_2) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_1 & 0 & a_2 & 0 & b_1 & 0 & b_2 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_3 & 0 & a_4 & 0 & b_3 & 0 & b_4 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & c_1 & 0 & c_2 & 0 & d_1 & 0 & d_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & c_3 & 0 & c_4 & 0 & d_3 & 0 & d_4 \end{bmatrix}.$$

We have the following key fact, which follows by a direct computation.

Lemma 3.2. *If $g_1 \in G_1$ and $g_2 \in G_2$, then $i(m(g_1, g_2), 1) = i_1' i_1(g_1, 1) i_2' i_2(g_2, 1)$.*

In the yet to be completed theory of doubling zeta integrals for the symplectic group, an important role will be played by integrals of good χ-sections over $i(U \times 1)$. The operator defined by this integration should send the space of good χ-sections for H onto the tensor product of the corresponding spaces for H_1 and H_2. Such a result should suffice to show that the zeta integral ideal of a representation of G parabolically induced from P is the product of the zeta integral ideals of the inducing data. Less knowledge, along with the fact that every element of $\mathrm{Irr}(G)$ can be embedded in a representation of G parabolically induced from a supercuspidal representation, allows one to prove the basic facts about zeta integrals stated in sections 1 and 2. Here, we prove only what we need, namely that formally the operator takes sections for H to the tensor product of sections for H_1 and H_2.

Lemma 3.3. *Let $v(x, y, z) \in V$, $p_1 \in P_1'$ and $p_2 \in P_2'$, with*

$$p_1 = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}, \quad p_2 = \begin{bmatrix} a_1 & a_2 & b_1 & b_2 \\ a_3 & a_4 & b_3 & b_4 \\ 0 & 0 & a_1' & a_2' \\ 0 & 0 & a_3' & a_4' \end{bmatrix}.$$

Then there exist $p' = \begin{bmatrix} a' & b' \\ 0 & {}^t a'^{-1} \end{bmatrix}, p'' = \begin{bmatrix} a'' & b'' \\ 0 & {}^t a''^{-1} \end{bmatrix}$ *in* P' *with* $\det a' = \det a'' = 1$ *and*

$$\begin{aligned} v(x,y,z)i_1'(p_1) &= i_1'(p_1)p'v(c^{-1}x, c^{-1}y{}^t c^{-1}, c^{-1}z), \\ v(x,y,z)i_2'(p_2) &= i_2'(p_2)p''v(xa_1 + za_3, y - y', xa_2 + za_4), \end{aligned}$$

where $y' = (xb_1 + zb_3)^t(xa_1 + za_3) + (xb_2 + zb_4)^t(xa_2 + za_4)$.

Proof. The lemma follows by a direct computation, with

$$a' = \begin{bmatrix} 1 & -a^{-1}bc^{-1}x & -a^{-1}bc^{-1}y{}^t c^{-1} & -a^{-1}bc^{-1}z \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, a'' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -{}^t(xb_1 + zb_3) & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -{}^t(xb_2 + zb_4) & 1 \end{bmatrix}$$

and

$$b' = \begin{bmatrix} -a^{-1}bc^{-1}y{}^t(a^{-1}bc^{-1}) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad b'' = 0. \quad \square$$

For the remainder of this paper, we let $\mu_1 = \chi|\cdot|^{(2n-k+1)/2}$ and $\mu_2 = \chi^{-1}|\cdot|^{(2n-k+1)/2}$. The following result will be used in the proof of Proposition 5.1.

Proposition 3.4. *Fix Haar measures as in Lemma 3.1. Let* S_1 *and* S_2 *be subsets of* H_1 *and* H_2*, respectively, such that* $p_1S_1 \subset S_1$ *and* $p_2S_2 \subset S_2$ *for* $p_1 \in P_1'$ *and* $p_2 \in P_2'$*. Let* S *be a subset of* H *such that* $i(U \times 1)i_1'(S_1)i_2'(S_2) \subset S$ *and* $pS \subset S$ *for* $p \in P'$*. Fix* $s \in \mathbb{C}$*, and let* $F : S \to [0, \infty]$ *be a function such that* $F(ph) = |\alpha_\chi(p, s)|F(h)$ *for* $p \in P'$ *and* $h \in S$*, and* $F(*h) \circ i|_{U\times 1}$ *is measurable for* $h \in i_1'(S_1)i_2'(S_2)$*. Define* $f : S_1 \times S_2 \to [0, \infty]$ *by*

$$f(h_1, h_2) = \int_U F(i(u, 1)i_1'(h_1)i_2'(h_2))\,du.$$

Then $f(p_1h_1, p_2h_2) = |\alpha_{\mu_1,\mu_2}(p_1, s)\alpha_\chi(p_2, s)|f(h_1, h_2)$ *for* $p_1 \in P_1'$*,* $p_2 \in P_2'$*,* $h_1 \in S_1$ *and* $h_2 \in S_2$*.*

Proof. The proof follows by a straightforward computation using Lemmas 3.1 and 3.3. $\square$

A similar argument proves the following. Suppose the Haar measures and S_1, S_2 and S are as in Proposition 3.4. Fix $s \in \mathbb{C}$, and let $F : S \to \mathbb{C}$ be a function such that $F(ph) = \alpha_\chi(p, s)F(h)$ for $p \in P'$ and $h \in S$, and for $h \in i_1'(S_1)i_2'(S_2)$, $F(*h) \circ i|_{U\times 1} \in L^1(U)$. Define $f : S_1 \times S_2 \to \mathbb{C}$ by

$$f(h_1, h_2) = \int_U F(i(u, 1)i_1'(h_1)i_2'(h_2))\,du.$$

Then $f(p_1h_1, p_2h_2) = \alpha_{\mu_1,\mu_2}(p_1, s)\alpha_\chi(p_2, s)f(h_1, h_2)$ for $p_1 \in P_1'$, $p_2 \in P_2'$, $h_1 \in S_1$ and $h_2 \in S_2$.

4. Generating standard χ-sections. In this section we recall a set of standard χ-sections that generate the space of standard χ-sections under the action of K'. These generators are well known; see Proposition 7.2.1 of [KR]. We will use these generators to construct the standard χ-section of the main theorem.

We need some more notation. Let m be a positive integer. Let $K(m)$ be the m-th principal congruence subgroup of K consisting of the $k \in K$ such that k is congruent to 1 mod $\mathfrak{P}^m$. Similarly define the m-th principal congruence subgroup $K'(m)$ of K'. Also, let $K'_0(m)$ be the subgroup of K' of elements whose lower left entries are congruent to 0 mod $\mathfrak{P}^m$. One can verify that $K'_0(m) = (K' \cap P')i(K(m) \times 1)$.

Given functions on G, we can construct candidates for standard χ-sections in the following way. Let ϕ be a function on G. Define $\Phi_\phi : H \times \mathbb{C} \to \mathbb{C}$ by

$$\Phi_\phi(h,s) = \begin{cases} 0 \text{ if } h \notin P'i(G \times 1) \\ \alpha_\chi(p,s)\phi(g) \text{ if } h \in P'i(G \times 1),\ h = pi(g,1). \end{cases}$$

Since $P' \cap i(G \times 1) = 1$, Φ_ϕ is well-defined. Clearly, $\Phi_\phi(ph,s) = \alpha_\chi(p,s)\Phi_\phi(h,s)$ for $p \in P'$, $h \in H$ and $s \in \mathbb{C}$.

Lemma 4.1. *Let m be a positive integer such that $m \geq \mathrm{c}(\chi)$, and let ϕ_m be the characteristic function of $K(m)$. Then Φ_{ϕ_m} is a standard χ-section, and the K' subspace of $\mathrm{I}_{2n}^{\mathrm{Stan}}(\chi)$ generated by the Φ_{ϕ_m} for $m \geq c(\chi)$ is $\mathrm{I}_{2n}^{\mathrm{Stan}}(\chi)$.*

Proof. Let $m \geq \mathrm{c}(\chi)$. Let $s \in \mathbb{C}$. The support of $\Phi_{\phi_m}(*,s)|_{K'}$ is $(P' \cap K')i(K(m) \times 1) = K'_0(m)$, and if $k \in K'_0(m)$ then $\Phi_{\phi_m}(k,s) = \chi(\det a)$; here, a is the upper left entry of k. It follows that $\Phi_{\phi_m}(kk',s) = \Phi_{\phi_m}(k,s)$ for $k \in K'$ and $k' \in K'(m)$. This implies that $\Phi_{\phi_m}(*,s) \in \mathrm{I}_{2n}(s,\chi)$. It is clear that for $k \in K'$, $\Phi_{\phi_m}(k,s)$ is independent of s. Thus, $\Phi_{\phi_m} \in \mathrm{I}_{2n}^{\mathrm{Stan}}(\chi)$.

Now let $\Phi \in \mathrm{I}_{2n}^{\mathrm{Stan}}(\chi)$. Let m be a positive integer such that $m \geq \mathrm{c}(\chi)$ and $k\Phi = \Phi$ for $k \in K'(m)$. Let $K' = \cup_{i=1}^t K'_0(m)k_i$ be a disjoint coset decomposition. Then $\Phi = \sum_{i=1}^t \Phi(k_i)(k_i^{-1}\Phi_{\phi_m})$. $\square$

There is an entirely similar construction of generators for $\mathrm{I}_{\mathrm{Gl}(2k)}^{\mathrm{Stan}}(\mu_1,\mu_2)$. We omit the details. The following result is well known and will be needed in the proof of the main theorem.

Proposition 4.2. *Let $\Phi \in \mathrm{I}_{2n}^{\mathrm{Stan}}(\chi)$. Then there exists a positive integer m such that $\Phi(i(kg,1),s) = \Phi(i(g,1),s)$ for $k \in K(m)$, $g \in G$ and $s \in \mathbb{C}$.*

Proof. Let m be a positive integer such that $\Phi(*,0)$ is right $K'(m)$ invariant. Let $g \in G$, and let $i(g,1) = p'k'$ with $p' \in P'$ and $k' \in K'$. Then for $k \in K(m)$ and $s \in \mathbb{C}$ we have $\Phi(i(kg,1),s) = \Phi(i(k^{-1},k^{-1})i(kg,1),s) = \alpha_\chi(p',s)\Phi(k'i(k^{-1},1),0) = \alpha_\chi(p',s)\Phi(k',0) = \Phi(i(g,1),s)$. $\square$

5. Reduction of χ-sections from H to H_1 and H_2. As mentioned in section 3, to generally relate the zeta integral ideal of a representation of G parabolically induced from P to the zeta integral ideals of the inducing data it will be necessary to investigate the operator which is defined by integration over $i(U \times 1)$. However, for Langlands quotients it seems necessary to consider integrals over $i(U \times 1)i(1 \times \overline{U})$, where $\overline{U}$ is the unipotent radical of the parabolic subgroup of G opposite to P. See the proof of Theorem 8.1, where we follow (3.5) of [J]. At the moment, we do not know how such integrals should be generally understood. In any case, in Proposition 5.1 we show that the generating sections from the last section behave well when integrated over the larger group. Since these generating sections cannot produce nontrivial zeta integrals, it is necessary to obtain the same result for sections that do yield nontrivial zeta integrals. Fortunately, the elements of $i(1 \times \overline{U})$ commute with a sufficiently robust subgroup of $i_1'(H_1)$, so that we can translate in Corollary 5.3 and obtain a version for sections that produce the desired nontrivial L-factor of Theorem 8.1.

Proposition 5.1. *Fix Haar measures on U and $\overline{U}$. Let m be a positive integer with $m \geq c(\chi)$. Let ϕ_m, ϕ_m^1 and ϕ_m^2 be the characteristic functions of $K(m)$, $K_1(m)$ and $K_2(m)$, respectively. Let ϕ_m' be ϕ_m divided by the volume of $(U \cap K(m)) \times (\overline{U} \times K(m))$. Let $\Phi_{\phi_m'} \in \mathrm{I}_{2n}^{\mathrm{Stan}}(\chi)$, $\Phi_{\phi_m^1} \in \mathrm{I}_{\mathrm{Gl}(2k)}^{\mathrm{Stan}}(\mu_1,\mu_2)$ and $\Phi_{\phi_m^2} \in \mathrm{I}_{2(n-k)}^{\mathrm{Stan}}(\chi)$ correspond to ϕ_m', ϕ_m^1 and ϕ_m^2 as in section 4. Then*

$$\int_{U\times\overline{U}} |\Phi_{\phi_m'}(i(u,1)i_1'(h_1)i_2'(h_2)i(1,\bar{u}),s)|\,d(u\bar{u}) < \infty$$

for all $h_1 \in H_1$, $h_2 \in H_2$ and $s \in \mathbb{C}$, and for such h_1, h_2 and s,

$$\Phi_{\phi_m^1}(h_1,s)\Phi_{\phi_m^2}(h_2,s) = \int_{U\times\overline{U}} \Phi_{\phi_m'}(i(u,1)i_1'(h_1)i_2'(h_2)i(1,\bar{u}),s)\,d(u\bar{u}),$$
$$|\Phi_{\phi_m^1}(h_1,s)\Phi_{\phi_m^2}(h_2,s)| = \int_{U\times\overline{U}} |\Phi_{\phi_m'}(i(u,1)i_1'(h_1)i_2'(h_2)i(1,\bar{u}),s)|\,d(u\bar{u}).$$

Proof. Fix $s \in \mathbb{C}$. First we prove the absolute convergence and equalities for $h_1 = i_1(g_1,1)$ and $h_2 = i_2(g_2,1)$ with $g_1 \in G_1$ and $g_2 \in G_2$. Let $u \in U$ and $\bar{u} \in \overline{U}$. Write

$$g_2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad u = \begin{bmatrix} 1 & x & y & z \\ 0 & 1 & {}^t z & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -{}^t x & 1 \end{bmatrix}, \quad \overline{u}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ {}^t\bar{x} & 1 & 0 & 0 \\ {}^t\bar{y} & \bar{z} & 1 & -\bar{x} \\ {}^t\bar{z} & 0 & 0 & 1 \end{bmatrix}.$$

Then by Lemma 3.2,

$$\Phi_{\phi_m'}(i(u,1)i_1'(h_1)i_2'(h_2)i(1,\bar{u}),s) = \Phi_{\phi_m'}(i(u,1)i(m(g_1,g_2),1)i(1,\bar{u}),s)$$

$$= \Phi_{\phi'_m}(i(\bar{u}^{-1}um(g_1,g_2)),1),s)$$
$$= \phi'_m(\bar{u}^{-1}um(g_1,g_2)).$$

Now $\bar{u}^{-1}um(g_1,g_2)$ is

$$\begin{bmatrix} g_1 & xa+zc & y\,{}^tg_1^{-1} & xb+zd \\ {}^t\bar{x}g_1 & a+{}^t\bar{x}(xa+zc) & ({}^tz+{}^t\bar{x}y){}^tg_1^{-1} & b+{}^t\bar{x}(xb+zd) \\ {}^t\bar{y}g_1 & \bar{z}a-\bar{x}c+{}^t\bar{y}(xa+zc) & (1+\bar{x}{}^tx+\bar{z}{}^tz+{}^t\bar{y}y){}^tg_1^{-1} & \bar{z}b-\bar{x}d+{}^t\bar{y}(xb+zd) \\ {}^t\bar{z}g_1 & c+{}^t\bar{z}(xa+zc) & (-{}^tx+{}^t\bar{z}y){}^tg_1^{-1} & d+{}^t\bar{z}(xb+zd) \end{bmatrix}.$$

We have $\phi'_m(\bar{u}^{-1}um(g_1,g_2))\neq 0$ if and only if $g_1\in K_1(m)$, $g_2\in K_2(m)$ and $u,\bar{u}\in K(m)$. This implies the proposition on $i_1(G_1\times 1)\times i_2(G_2\times 1)$.

Next, we prove the proposition on $S_1\times S_2$, with $S_1=P'_1i_1(G_1\times 1)$, $S_2=P'_2i_2(G_2\times 1)$. We will use Proposition 3.4. Let $S=P'i(U\times 1)i'_1(S_1)i'_2(S_2)$. Define $F:S\to[0,\infty]$ by

$$F(h)=\int_{\overline{U}}|\Phi_{\phi'_m}(hi(1,\bar{u}),s)|\,d\bar{u}.$$

We assert that $F(h)<\infty$ for $h\in S$. It suffices to show this for $h\in i(U\times 1)i'_1(S_1)i'_2(S_2)$. Let $h=i(u,1)i'_1(p_1i_1(g_1,1))i'_2(p_2i_2(g_2,1))$, where $u\in U$, $p_1\in P'_1$, $p_2\in P'_2$, $g_1\in G_1$ and $g_2\in G_2$. By Lemma 3.3 and the identity before Lemma 3.1, $i(u,1)i'_1(p_1)i'_2(p_2)=p'i(u',1)$ for some $p'\in P'$ and $u'\in U$. If $\bar{u}\in\overline{U}$, then $\Phi_{\phi'_m}(hi(1,\bar{u}),s)=\alpha_\chi(p',s)\phi'_m(\bar{u}^{-1}u'm(g_1,g_2))$, which implies that $\bar{u}\mapsto\Phi_{\phi'_m}(hi(1,\bar{u}),s)$ has compact support in $\overline{U}$, proving our assertion. Now F satisfies the hypotheses of Proposition 3.4, and by the last paragraph and Proposition 3.4 we see that for $g_1\in G_1$, $g_2\in G_2$, $p_1\in P'_1$ and $p_2\in P'_2$,

$$\int_{U\times\overline{U}}|\Phi_{\phi'_m}(i(u,1)i'_1(p_1i_1(g_1,1))i'_2(p_2i_2(g_2,1))i(1,\bar{u}),s)|\,d(u\bar{u})$$

is $|\Phi_{\phi^1_m}(p_1i_1(g_1,1),s)\Phi_{\phi^2_m}(p_2i_2(g_2,1),s)|$. It follows that the integral is finite, as claimed, and that the second equality of the proposition holds on $S_1\times S_2$. The first equality follows by a similar argument, using the remark after Proposition 3.4.

Finally, suppose that $h_1\in H_1$ and $h_2\in H_2$, but $h_1\notin P'_1i_1(G_1\times 1)$ or $h_2\notin P'_2i_2(G_2\times 1)$. Since $\Phi_{\phi^1_m}(h_1)\Phi_{\phi^2_m}(h_2)=0$, it will suffice to show that the integrand of our integral is identically zero. Since $P'_1i_1(G_1\times 1)$ and $P'_2i_2(G_2\times 1)$ are dense in H_1 and H_2, respectively, there exist sequences $\{h_1(n)\}=\{p_1(n)i_1(g_1(n),1)\}$ and $\{h_2(n)\}=\{p_2(n)i_2(g_2(n),1)\}$ such that $p_1(n)\in P'_1$, $p_2(n)\in P'_2$, $g_1(n)\in G_1$, $g_2(n)\in G_2$ and $h_1(n)\to h_1$ and $h_2(n)\to h_2$. Assume that $g_1(n)\in K_1(m)$ for only finitely many n. For any n such that $g_1(n)\notin K_1(m)$, by the last paragraph

$$\int_{U\times\overline{U}}|\Phi_{\phi'_m}(i(u,1)i'_1(h_1(n))i'_2(h_2(n))i(1,\bar{u}),s)|\,d(u\bar{u})$$

is $|\Phi_{\phi^1_m}(p_1(n)i_1(g_1(n),1))\Phi_{\phi^2_m}(p_2(n)i_2(g_2(n),1))| = 0$. Since $\Phi_{\phi'_m}$ is continuous, this implies that $\Phi_{\phi'_m}(i(u,1)i'_1(h_1(n))i'_2(h_2(n))i(1,\bar{u}),s) = 0$ for $u \in U$ and $\bar{u} \in \overline{U}$ for all n such that $g_1(n) \notin K_1(m)$. Taking limits, we get $\Phi_{\phi'_m}(i(u,1)i'_1(h_1)i'_2(h_2)i(1,\bar{u}),s) = 0$ for all $u \in U$ and $\bar{u} \in \overline{U}$, proving our claim. Suppose that $g_1(n) \in K_1(m)$ for infinitely many n. Passing to a subsequence, we may assume that $g_1(n) \in K_1(m)$ for all n, and by a similar argument, that $g_2(n) \in K_2(m)$ for all n. Since $K_1(m)$ and $K_2(m)$ are compact, we may assume that $g_1(n) \to k_1 \in K_1(m)$ and $g_2(n) \to k_2 \in K_2(m)$. This implies that $\{p_1(n)\}$ and $\{p_2(n)\}$ converge to elements p_1 and p_2 of P'_1 and P'_2, respectively. Hence, $h_1 = p_1 i_1(k_1,1)$ and $h_2 = p_2 i_2(k_2,1)$, contradicting our hypothesis. □

Now we translate, as mentioned at the beginning of this section. In fact, there are other elements of $i'(H_1)$ that centralize $i(1 \times \overline{U})$ and which would serve our purposes.

Lemma 5.2. *The element* $i'_1(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix})$ *of* H *is contained in the centralizer in* H *of* $i(1 \times \overline{U})$.

Proof. This follows by a direct computation. □

Corollary 5.3. *Let the notation be as in Proposition 5.1, and let*

$$\Phi_0 = i'_1(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix})\Phi_{\phi'_m}, \quad \Phi_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\Phi_{\phi^1_m}, \quad \Phi_2 = \Phi_{\phi^2_m}.$$

Then Φ_0 *and* Φ_1 *are standard, and the conclusion of Proposition 5.1 holds, with* Φ_0, Φ_1 *and* Φ_2 *in place of* $\Phi_{\phi'_m}, \Phi_{\phi^1_m}$ *and* $\Phi_{\phi^2_m}$, *respectively.*

6. A Gl(1) **zeta integral.** In this section, in the case $k = 1$, we compute the Gl(1) zeta integrals associated to the section Φ_1 defined at the end of the last section. We obtain an L-factor of the same form as the L-factor of Theorem 8.1, and to prove Theorem 8.1, we will show that there is a zeta integral of π that is equal to an appropriate Gl(1) zeta integral associated to Φ_1. In this section, we assume $k = 1$. The twist of β by $|\cdot|^{-(2n-k+1)/2}$ in the following proposition compensates for the twist by $|\cdot|^{(2n-k+1)/2}$ in the definitions of μ_1 and μ_2 which arose naturally in Proposition 3.4; see also the proof of Theorem 8.1.

Proposition 6.1. *Let* m *be a positive integer with* $m \geq c(\chi)$, *and let* Φ_1 *be defined as in Corollary 5.3. Let* $\beta \in \mathrm{Irr}(\mathrm{Gl}(1,F))$, *i.e., let* β *be a quasi-character of* $F^\times$. *Then for some nonzero constants* A *and* B,

$$Z(s, \beta \otimes |\cdot|^{-(2n-k+1)/2}, \Phi_1) = A(\int_{\mathfrak{O}^\times} \beta\chi^{-1}(u)\,du) q^{Bs} L(s+1/2, \chi\beta^{-1}).$$

Proof. Let $s \in \mathbb{C}$ and $h \in F^\times$, $h \neq 1$. Then

$$i_1(h,1)\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & h \\ 0 & 1-h \end{bmatrix} i_1(h',1),$$

where $h' = h/(h-1)$. So $\Phi_1(i_1(h,1),s) = \chi(1-h)^{-1}|1-h|^{-s-1/2+n}\phi_m^1(h')$. Now $h' \in K_1(m)$ if and only if $|h| \geq |\pi_F|^{-m}$. Also, if $|h| \geq |\pi_F|^{-m}$, then $\chi(1-h)^{-1}|1-h|^{-s-1/2+n} = \chi(-1)\chi(h)^{-1}|h|^{-s-1/2+n}$. If $h=1$ then $1 = i_1(h,1)$ is not in the support of Φ_1. Hence,

$$\Phi_1(i_1(h,1),s) = \begin{cases} \chi(-1)\chi(h)^{-1}|h|^{-s-1/2+n} \text{ if } |h| \geq |\pi_F|^{-m} \\ 0 \text{ if } |h| < |\pi_F|^{-m}. \end{cases}$$

A direct computation of $Z(s, \beta \otimes |\cdot|^{-(2n-k+1)/2}, \Phi_1)$ now completes the proof. □

7. Matrix coefficients. In this section we recall some facts about Langlands quotients and their matrix coefficients. To prove Theorem 8.1 we will need to construct some matrix coefficients of π, and to do so we will use the same method as in the case of the general linear group. Thus, we need the analogues of (3.3.4)-(3.3.8) of [J]. These results are well-known, and we state them without proofs. As a general reference for Langlands quotients we mention [BW], and for the case of the symplectic group, [T], section 6.

The following notation will be fixed throughout this section. Let $n = n_1 + \cdots + n_t + n_0$ be an ordered partition of n with $n_1, \ldots, n_t$ positive if $t > 0$. Let $\delta_i \in \mathrm{Irr}(\mathrm{Gl}(n_i, F))$, $1 \leq i \leq t$, be essentially tempered with $e(\delta_1) > \cdots > e(\delta_t) > 0$, and let $\tau \in \mathrm{Irr}(\mathrm{Sp}(n_0, F))$ be tempered. Let $\sigma = \delta_1 \otimes \cdots \otimes \delta_t \otimes \tau$. Let $Q = P'_{n_1,\ldots,n_t}$ be the parabolic subgroup of $G = \mathrm{Sp}(n, F)$ with Levi factor L isomorphic to $\mathrm{Gl}(n_1, F) \times \cdots \times \mathrm{Gl}(n_t, F) \times \mathrm{Sp}(n_0, F)$, as defined in [R1]. Let $V = U'_{n_1,\ldots,n_t}$ be the unipotent radical of Q. It is known that $\mathrm{Ind}_Q^G \sigma$ has a unique irreducible quotient π, called the Langlands quotient.

The following proposition characterizes the coefficients of π.

Proposition 7.1. *Let $H : G \times G \to \mathbb{C}$ be a function such that*

1. *$H(vmg, \bar{v}mg') = H(g, g')$ for $v \in V$, $\bar{v} \in \overline{V}$, $m \in L$, and $g, g' \in G$;*
2. *For all $g, g' \in G$, the function from L to $\mathbb{C}$ that sends m to $H(mg, g')$ is a coefficient of $\sigma \otimes \delta_Q^{1/2}$;*
3. *There exists a compact open subgroup $\Gamma \subset K$ such that $H(g\gamma, g'\gamma') = H(g, g')$ for $g, g' \in G$ and $\gamma, \gamma' \in \Gamma$.*

Then there exists a positive integer r and $f_i \in \mathrm{Ind}_Q^G \sigma$ and $f'_i \in \mathrm{Ind}_{\overline{Q}}^G \sigma^\vee$ for $1 \leq i \leq r$ such that for $g, g' \in G$, $H(g, g') = \sum_{i=1}^r \langle f_i(g), f'_i(g') \rangle_\sigma$. Here, $\langle \cdot, \cdot \rangle_\sigma$ is the canonical pairing $\sigma \otimes \sigma^\vee \to \mathbb{C}$. Moreover, if Haar measures on $\overline{V}$ and K are fixed, then for each $g \in G$,

$$\int_{\overline{V} \times K} H(\bar{v}kg, k)\, d(vk),$$

converges absolutely, and the function on G to $\mathbb{C}$ that sends g to this integral is a coefficient of π. Conversely, every coefficient of π is of this form.

It will be useful to have a slight modification of Proposition 7.1. Let P be the maximal parabolic denoted by P'_{n_1} in [R1], with Levi factor M isomorphic to $\mathrm{Gl}(n_1, F) \times \mathrm{Sp}(n', F)$, where $n' = n_2 + \cdots + n_t + n_0$; this is the same as the parabolic P of section 3, with $k = n_1$. Let U be the unipotent radical of P. Let π' be the Langlands quotient of $\mathrm{Ind}_{P'_{n_2,\ldots,n_t}}^{\mathrm{Sp}(n',F)}(\delta_2 \otimes \cdots \otimes \delta_t \otimes \tau)$, and let $\sigma' = \delta_1 \otimes \pi'$.

Proposition 7.2. *The statement of Proposition 7.1 holds, with σ, Q, L and V replaced by σ', P, M and U respectively.*

8. Proofs of the main results. As we indicated in the introduction, Theorem 8.2 will follow from Theorem 4.4 of [R1] and Theorem 8.1. We begin by proving Theorem 8.1. The argument is very similar to the proof of Theorem 3.4 of [J]. In this section we assume that $k = 1$, though this is used only as a hypothesis for Proposition 6.1.

Proof of Theorem 8.1. Let Φ_0, Φ_1 and Φ_2 be as in Corollary 5.3. Let the notation be as in section 7 and the statement of Theorem 8.1. To prove Theorem 8.1 we will proceed as follows. Fix a coefficient f_2 for π' such that $f_2(1) \neq 0$. As is pointed out in [KR], and as is easy to see, if m is sufficiently large, then $Z(s, \Phi_2, f_2)$ is nonzero and constant for all $s \in \mathbb{C}$. Fix such an m. Let $f_1 = \delta_1 \otimes |\cdot|^{-(2n-k+1)/2}$. The twist by $|\cdot|^{-(2n-k+1)/2}$ is natural: see the comment at the beginning of section 6 and the construction of H below. To prove Theorem 8.1, by Proposition 6.1, it will suffice to construct a coefficient f_0 of π such that $Z(s, f_0, \Phi_0) = Z(s, f_1, \Phi_1)Z(s, f_2, \Phi_2)$. To construct f_0, we will follow the idea from the proof of Theorem 3.4 of [J]. To simplify the notation, we will write (g, g') for $i(g, g')$, etc.

To begin, we invoke Corollary 5.3. The integrals defining $Z(s, \Phi_1, f_1)$ and $Z(s, \Phi_2, f_2)$ converge absolutely for $\mathrm{Re}(s)$ sufficiently large. Fix s with $\mathrm{Re}(s)$ large. Then

$$Z(s, \Phi_1, f_1)Z(s, \Phi_2, f_2) = \int\limits_{G_1 \times G_2} \Phi_1(g_1, 1)\Phi_2(g_2, 1)F(g_1, g_2)\, d(g_1 g_2),$$

where the convergence is absolute, and to simplify the notation, we write $F(g_1, g_2) = f_1(g_1)f(g_2)$ for $g_1 \in G_1$ and $g_2 \in G_2$, and $\Phi_1(g, 1)$ for $\Phi_1((g, 1), s)$ and $\Phi_2(g, 1)$ for $\Phi_2((g, 1), s)$. By Lemma 3.2 and Corollary 5.3,

$$Z(s, f_1, \Phi_1)Z(s, f_2, \Phi_2) = \int\limits_{G_1 \times G_2 \times U \times \overline{U}} \Phi_0(\bar{u}^{-1}um(g_1, g_2), 1)F(g_1, g_2)\, d(g_1 g_2 u\bar{u}),$$

where the convergence is absolute.

Now we use Proposition 4.2 to introduce an extra integration over $K \times K$. Fix a Haar measure on K. By Proposition 4.2, the restriction of Φ_0 to $G \times 1$ is left and right invariant under $K(l)$ for some positive integer l. Let η be the characteristic function of $K(l) \times K(l)$ divided by the volume of $K(l) \times K(l)$. Then for $g \in G$,

$$\Phi(g, 1) = \int\limits_{K \times K} \Phi_0(k^{-1}gk', 1)\eta(k, k')\, d(kk'),$$

so that $Z(s, f_1, \Phi_1)Z(s, f_2, \Phi_2)$ is

$$\int\limits_{K \times K} \Big[\int\limits_{G_1 \times G_2 \times U \times \overline{U}} \Phi_0((\bar{u}k)^{-1}um(g_1, g_2)k', 1)F(g_1, g_2)\eta(k, k')\, d(g_1 g_2 u\bar{u})\Big]\, d(kk').$$

Here, the inner integral converges absolutely for all $k, k' \in K$, and the function on $K \times K$ that sends (k, k') to the inner integral is locally constant, hence continuous.

Next, we decompose the outer integral to obtain a function H as in Proposition 7.2. Let $L = K \cap \overline{P}$, $L' = K \cap P$. Then L and L' are compact and closed in K. Fix Haar measures on L and L'. There exist unique right K invariant quotient measures on $L\backslash K$ and $L'\backslash K$ such that $Z(s, f_1, \Phi_1)Z(s, f_2, \Phi_2)$ is

$$\int\limits_{L\backslash K\times L'\backslash K} [\int\limits_{L\times L'} [\int\limits_{G_1\times G_2\times U\times\overline{U}} \Phi_0((\bar{u}hk)^{-1}um(g_1, g_2)h'k', 1)F(g_1, g_2)\, d(g_1g_2u\bar{u})] \\ \eta(hk, h'k')\, d(hh')]\, d(kk').$$

We consider the inner integral. Fix $k, k' \in K$ and $h \in L$ and $h' \in L'$. Write $h = \bar{u}(h)m(h)$ and $h' = u(h')m(h')$, where $\bar{u}(h) \in \overline{U}$, $u(h') \in U$, $m(h), m(h') \in M$. Let $h_1 \in K_1$ and $h_2 \in K_2$ be such that $m(h_1, h_2) = m(h)$; similarly define h_1' and h_2'. A computation shows that

$$\int\limits_{G_1\times G_2\times U\times\overline{U}} \Phi_0((\bar{u}hk)^{-1}um(g_1, g_2)h'k', 1)F(g_1, g_2)\, d(g_1g_2u\bar{u}) \\ = \int\limits_{G_1\times G_2\times U\times\overline{U}} \delta_P(m(g_1, g_2))\Phi_0((\bar{u}k)^{-1}m(g_1, g_2)uk', 1)F(h_1g_1{h_1'}^{-1}, h_2g_2{h_2'}^{-1})\, d(g_1g_2u\bar{u}),$$

where both integrals converge absolutely. We now have

$$Z(s, f_1, \Phi_1)Z(s, f_2, \Phi_2) = \int\limits_{L\backslash K\times L'\backslash K} [\int\limits_{L\times L'} [\int\limits_{G_1\times G_2\times U\times\overline{U}} \delta_P(m(g_1, g_2)) \\ \Phi_0((\bar{u}k)^{-1}m(g_1, g_2)uk', 1)F(h_1g_1{h_1'}^{-1}, h_2g_2{h_2'}^{-1})\, d(g_1g_2u\bar{u})]\eta(hk, h'k')\, d(hh')]\, d(kk').$$

Since for any $k, k' \in K$ the inner double integral converges absolutely, we have that $Z(s, f_1, \Phi_1)Z(s, f_2, \Phi_2)$ is

$$\int\limits_{L\backslash K\times L'\backslash K} [\int\limits_{G_1\times G_2\times U\times\overline{U}} H_1(m(g_1, g_2), k, k')\Phi_0((\bar{u}k)^{-1}m(g_1, g_2)uk', 1)\, d(g_1g_2u\bar{u})]\, d(kk'),$$

where

$$H_1(m(g_1, g_2), k, k') = \delta_P(m(g_1, g_2)) \int\limits_{L\times L'} F(h_1g_1{h_1'}^{-1}, h_2g_2{h_2'}^{-1})\eta(hk, h'k')\, d(hh').$$

Define $H : G \times G \to \mathbb{C}$ by $H(umk, \bar{u}m'k') = H_1(m'^{-1}m, k', k)$ for $u \in U$, $\bar{u} \in \overline{U}$, $m, m' \in M$, and $k, k' \in K$. Then computations show that H is well-defined and H satisfies hypotheses (1), (2) and (3) of Proposition 7.2, which we will apply at the end of the proof. In particular, note that the twist of δ_1 by $|\cdot|^{-(2n-r+1)/2}$ in the definition of f_1 is required for (2).

We now have that $Z(s, f_1, \Phi_1)Z(s, f_2, \Phi_2)$ is

$$\int_{L\backslash K \times L'\backslash K} [\int_{G_1 \times G_2 \times U \times \overline{U}} H(m(g_1, g_2)k', k)\Phi_0((\bar{u}k)^{-1}m(g_1, g_2)uk', 1)\, d(g_1g_2u\bar{u})]\, d(kk')$$

$$= c \int_{K \times K} [\int_{G_1 \times G_2 \times U \times \overline{U}} H(m(g_1, g_2)k', k)\Phi_0((\bar{u}k)^{-1}m(g_1, g_2)uk', 1)\, d(g_1g_2u\bar{u})]\, d(kk')$$

$$= c \int_{K \times K \times \overline{U}} [\int_{G_1 \times G_2 \times U} H(m(g_1, g_2)k', k)\Phi_0((\bar{u}k)^{-1}m(g_1, g_2)uk', 1)\, d(g_1g_2u)]\, d(kk'\bar{u}),$$

where c is $\mathrm{vol}(L)^{-1}\,\mathrm{vol}(L')^{-1}$. Now for all $k, k' \in K$ and $\bar{u} \in \overline{U}$, the inner integral converges absolutely, and for fixed $k, k' \in K$ and $\bar{u} \in \overline{U}$,

$$\int_{G_1 \times G_2 \times U} H(m(g_1, g_2)k', k)\Phi_0((\bar{u}k)^{-1}m(g_1, g_2)uk', 1)\, d(g_1g_2u)$$

$$= \int_{G_1 \times G_2} [\int_U \delta_P(m(g_1, g_2))^{-1} H(um(g_1, g_2)k', k)\Phi_0((\bar{u}k)^{-1}um(g_1, g_2)k', 1)\, du]\, d(g_1g_2)$$

$$= \int_P H(pk', k)\Phi_0((\bar{u}k)^{-1}pk', 1)\, d_lp,$$

where d_lp is the left Haar measure on P determined by the Haar measures on M and U. Thus,

$$Z(s, f_1, \Phi_1)Z(s, f_2, \Phi_2) = c \int_{K \times K \times \overline{U}} [\int_P H(pk', k)\Phi_0((\bar{u}k)^{-1}pk', 1)\, d_lp]\, d(kk'\bar{u}).$$

By the same computations with absolute values, we have

$$\int_{K \times K \times \overline{U}} [\int_P |H(pk', k)\Phi_0((\bar{u}k)^{-1}pk', 1)|\, d_lp]\, d(kk'\bar{u}) < \infty.$$

Hence,

$$Z(s, f_1, \Phi_1)Z(s, f_2, \Phi_2) = c \int_{K \times K \times \overline{U} \times P} H(pk', k)\Phi_0((\bar{u}k)^{-1}pk', 1)\, d(kk'\bar{u}p)$$

$$= c \int_{K\times\overline{U}} [\int_K [\int_P H(pk',k)\Phi_0((\bar{u}k)^{-1}pk',1)\,d_l p]\,dk']\,d(k\bar{u})$$
$$= c \int_{K\times\overline{U}} [\int_G H(g,k)\Phi_0((\bar{u}k)^{-1}g,1)\,dg]\,d(k\bar{u})$$
$$= c \int_{K\times\overline{U}} [\int_G H(\bar{u}kg,k)\Phi_0(g,1)\,dg]\,d(k\bar{u}),$$

where dg is the Haar measure on G determined by the Haar measures on P and K via the Iwasawa decomposition $G = PK$. Again, by the same computations with absolute values, we obtain

$$\int_{K\times\overline{U}} [\int_G |H(\bar{u}kg,k)\Phi_0(g,1)|\,dg]\,d(k\bar{u}) < \infty.$$

So,

$$Z(s,f_1,\Phi_1)Z(s,f_2,\Phi_2) = c \int_{K\times\overline{U}\times G} H(\bar{u}kg,k)\Phi_0(g,1)\,d(k\bar{u}g)$$
$$= c\int_G \Phi_0(g,1)[\int_{K\times\overline{U}} H(\bar{u}kg,k)\,d(k\bar{u})]\,dg.$$

The theorem now follows from Proposition 7.2 and our initial remarks. □

Proof of Theorem 8.2. We have $\pi = L(\chi_V|\cdot|^{n-l/2}\otimes\chi_V|\cdot|^{n-1-l/2}\otimes\cdots\otimes\chi_V|\cdot|^{n_0+1-l/2}\otimes\pi_0)$ by Theorem 4.4 of [R1]. The theorem now follows by applying Theorem 8.1 with $\chi = \chi_V$. □

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Department of Mathematics, University of Toronto, 100 St. George Street, Toronto, ON M5S 3G3, CANADA

E-mail address: `brooks@math.toronto.edu`