THE THETA CORRESPONDENCE FOR SIMILITUDES

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Abstract. In this paper we investigate the theta correspondence for similitudes over a nonarchimedean local field. We show that the two main approaches to a theta correspondence for similitudes from the literature are essentially the same, and we prove that a version of strong Howe duality holds for both constructions.

Suppose that k is a nonarchimedean local field, X is a finite dimensional nondegenerate symmetric bilinear space over k, and Y is a finite dimensional nondegenerate symplectic bilinear space over k. Let p be the projection from the metaplectic cover of $\mathrm{Sp}(X \otimes_k Y)$ to $\mathrm{Sp}(X \otimes_k Y)$. Fix a nontrivial additive character of k, and let r be the corresponding smooth Weil representation of the metaplectic cover of $\mathrm{Sp}(X \otimes_k Y)$. Then the restriction of r to $p^{-1}(\mathrm{O}(X))p^{-1}(\mathrm{Sp}(Y))$ defines a correspondence between the smooth admissible duals of $p^{-1}(\mathrm{O}(X))$ and $p^{-1}(\mathrm{Sp}(Y))$. When the residual characteristic of k is odd, this correspondence satisfies strong Howe duality. Conceivably, r might be used to construct a representation that involves similitudes of X and Y, so that a correspondence between the smooth admissible duals of some covers of $\mathrm{GO}(X)$ and $\mathrm{GSp}(Y)$ could be defined and analyzed. We consider two such constructions. The first extends the restriction of r to $p^{-1}(\mathrm{O}(X))p^{-1}(\mathrm{Sp}(Y))$ to a representation ω of a larger group involving similitudes. Apparently, ω was first implicitly introduced in [S], and first explicitly considered in [HK]. The second construction induces the restriction of r to $p^{-1}(\mathrm{Sp}(Y))$ to obtain a representation Ω that involves similitudes. As far as we know, Ω first appeared in [SA] in the case of finite fields, and in [PSS] in the case of nonarchimedean local fields. In this paper we show that the two approaches are essentially the same, and that when the residual characteristic of k is odd, a natural version of strong Howe duality holds for the associated correspondence.

The two constructions of a correspondence for similitudes already have proven to be valuable tools in automorphic representation theory and its applications. Many examples have been considered. As a sample, see [JL], [S], [Co], [So], [HK], and [HST]. In particular,

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the correspondence for similitudes should be useful in the investigation of Shimura varieties. It is similitudes that occur in the theory of Shimura varieties, not isometries.

To give a detailed account of our results we need more notation. Let $H = \mathrm{GO}(X)$. If $\dim_k X$ is even, let $G' = \mathrm{GSp}(Y)$. If $\dim_k X$ is odd, let G' be a certain two-fold cover of $\mathrm{GSp}(Y)$. For $g \in G'$, let $\lambda(g)$ be the similitude factor of the projection of g to $\mathrm{GSp}(Y)$. Similarly, let $\lambda(h)$ be the similitude factor of $h \in H$. Let G be the subgroup of $g \in G'$ such that $\lambda(g) \in \lambda(H)$. Also, let G_1 and H_1 be the subgroups of $g \in G$ and $h \in H$ such that $\lambda(g) = 1$ and $\lambda(h) = 1$, respectively. Then ω is a representation of the group

$$R = \{(g, h) \in G' \times H \colon \lambda(g) = \lambda(h)\},$$

and Ω is a representation of $G' \times H$. See sections 2 and 3 for precise definitions.

In section 1, following [R], [MVW] and [KR], we define Howe duality, multiplicity preservation, and strong Howe duality. In Proposition 1.1 we show that, taken together, Howe duality and multiplicity preservation are equivalent to strong Howe duality. This result is well known to experts. We also state another result that is used in the last section.

In sections 2 and 3 we carefully construct and relate ω and Ω. After recalling some basic facts about the metaplectic covers of similitude groups and splittings that follow from [Ra], [B] and [K], we show that the definitions of ω and Ω depend on the same fundamental identity in the metaplectic cover. Using the identity, we also prove that Ω is obtained from ω via compact induction:

$$\Omega \cong \text{c-Ind}_R^{G' \times H} \omega.$$

This is the first main result.

In section 4 we study the correspondence defined by ω, assuming that strong Howe duality holds for the usual correspondence for isometries. After the key observation that R only involves G, we investigate whether the condition

$$\mathrm{Hom}_R(\omega, \pi \otimes_{\mathbb{C}} \tau) \neq 0$$

for $\pi \in \mathrm{Irr}(G)$ and $\tau \in \mathrm{Irr}(H)$ gives rise to a correspondence satisfying the analogues of Howe duality and multiplicity preservation. Given that π and τ correspond, we show in Lemma 4.2 that the equivalence classes of irreducible constituents of $\pi|_{G_1}$ and $\tau|_{H_1}$ are paired via the usual correspondence for isometries, so that in particular the numbers of equivalence classes are the same; we also show that $\pi|_{G_1}$ is multiplicity free if and only if $\tau|_{H_1}$ is multiplicity free. This suggests that we restrict attention to representations with multiplicity free restrictions to G_1 and H_1, which we do. Then in Theorem 4.4 we prove that the analogues of Howe duality and multiplicity preservation hold. This is the second main result.

In sections 5 and 6 we consider the consequences of section 4 for Ω. First, we define a natural $G \times H$ subrepresentation Ω^+ of Ω such that

$$\Omega^+ \cong \text{c-Ind}_R^{G \times H} \omega.$$

We prove that strong Howe duality holds for Ω^+ with respect to the multiplicity free elements of $\mathrm{Irr}(G)$ and $\mathrm{Irr}(H)$ when strong Howe duality holds for the usual correspondence for isometries. Next, we consider whether the condition

$$\mathrm{Hom}_{G' \times H}(\Omega, \pi \otimes_{\mathbb{C}} \tau) \neq 0$$

for $\pi \in \mathrm{Irr}(G')$ and $\tau \in \mathrm{Irr}(H)$ defines a correspondence satisfying Howe duality in the case $G \neq G'$. When $G \neq G'$, $\dim_k X$ is even and the residual characteristic of k is odd, using that Ω^+ satisfies Howe duality, we give an equivalent condition based on Proposition 1.2. This condition is called the theta dichotomy in [HKS]. Using the condition, we show in this case that Howe duality does not hold for Ω in the stable range. We also point out that when $\dim_k X \leq \dim_k Y$, that is, when the theta dichotomy is expected to hold, strong Howe duality for Ω is expected to hold. This is the final main result.

The main previous general work in this area is, as far as we know, [B]. However, the approaches we consider, and the approach of [B], are different. The paper [B] investigates whether the compactly induced representation of r to the metaplectic cover of $\mathrm{GSp}(X \otimes_k Y)$, and the inverse images of $\mathrm{GO}(X)$ and $\mathrm{GSp}(Y)$ in the metaplectic cover are analogues of r, $p^{-1}(\mathrm{O}(X))$ and $p^{-1}(\mathrm{Sp}(Y))$. Complications arise since the inverse images of $\mathrm{GO}(X)$ and $\mathrm{GSp}(Y)$ are poor analogues of $p^{-1}(\mathrm{O}(X))$ and $p^{-1}(\mathrm{Sp}(Y))$. In particular, the elements of the inverse images of $\mathrm{GO}(X)$ and $\mathrm{GSp}(Y)$ do not always commute. As a consequence, [B] defines and considers a correspondence between sets of representations of the inverse images of $\mathrm{GO}(X)$ and $\mathrm{GSp}(Y)$ rather than representations. What we call ω and Ω are not what are called ω and Ω in [B]. Still, we draw heavily on [B] for results about the metaplectic group for similitudes.

We hope these results will be useful in several ways. The simple connection between ω and Ω should allow results about one representation to be applied to the other. As in [R], in the case of isometries, Howe duality and multiplicity preservation for ω will have fundamental consequences for the global correspondence of automorphic representations defined by ω. The results of section 4 are actually more general then stated above. The abstraction of section 4 may be useful in other contexts. These results may be useful in understanding examples in the literature.

We believe that the hypotheses for these results are not too restrictive for many applications. Let us consider some of the hypotheses. First, Howe duality and multiplicity preservation for ω require that Howe duality and multiplicity preservation hold for the correspondence for isometries. By a theorem of J.-L. Waldspurger [W] this requirement is satisfied if the residual characteristic of k is odd. It is conjectured to hold when the residual characteristic even. Second, in the above description of Howe duality and multiplicity preservation for ω we need that representations have multiplicity free restrictions to G_1 and H_1. In applications, one often begins with a representation of G or H and assumes or proves that there exists a representation of the other group so that the two representations correspond with respect to ω. By Lemma 4.2, if the initial representation has multiplicity free restriction, then so will the corresponding representation: having multiplicity free restriction is contagious. Thus, in the case when the initial representation has multiplicity free restriction, Howe duality and multiplicity preservation may be applied. Often, one can verify that the initial representation has multiplicity free restriction. For example, when $\dim_k X$ is two or four, cases that have many important applications beginning with [JL] and [S], all elements of $\mathrm{Irr}(H)$ have multiplicity free restrictions [HPS]. Also, if the initial representation is generic then it has multiplicity free restriction. On the other hand, if the initial representation does not have multiplicity free restriction, then by Lemma 4.2 neither will the corresponding representation. In this way new examples of representations without multiplicity free restrictions might be constructed.

Finally, we make some remarks about Ω. For the purposes of this paper, it seems that ω is more natural than Ω. In contrast to ω, the definition of Ω gives no indication that Howe duality should not always hold, or that it should hold for Ω^+. However, in other contexts it may be useful to consider Ω. One example might be seesaw reciprocity. Moreover, Ω has a definition independent of ω. This definition gives the so called extra variable Schrödinger model, which has been useful in some situations. The representations Ω and Ω^+ are not artificial objects.

The problem of Howe duality for Ω is not completely solved. This problem is fundamental and important. Many applications involve G'. See, for example, [HST], where the case $\dim_k X = \dim_k Y = 4$ is used. There the correspondence between $\mathrm{Irr}(G')$ and $\mathrm{Irr}(H)$ defined by ω is employed. By our results relating ω and Ω, this is equivalent to the consideration of the correspondence between $\mathrm{Irr}(G')$ and $\mathrm{Irr}(H)$ defined by Ω. Also, the problem is deep. As mentioned above, in the case $\dim_k X$ is even, G is a proper subgroup of G' and $\dim_k X \le \dim_k Y$, Howe duality for Ω is equivalent to theta dichotomy.

I would like to thank S.S. Kudla for his generous help and useful comments. In particular, he kindly allowed me access to his preprints [K], [KR] and [HKS], and it was his suggestion that Proposition 3.5 might be true.

We use the following notation. Let J be a group of td-type, as in [C]. This means that J is a topological group and every neighborhood of the identity element of J contains a compact open subgroup. We will often explicitly assume that J has a countable basis. In this case, Shur's lemma holds. See [C]. Let $\mathrm{Irr}(J)$ be the set of equivalence classes of smooth admissible irreducible representations of J. If $\pi \in \mathrm{Irr}(J)$ then $\pi^\vee \in \mathrm{Irr}(J)$ is the contragredient representation of π. A character of J is a continuous homomorphism from J to $\mathbb{C}^\times$. The notation for induction will be as in section 1.8 of [C]. In section 4 we will also use the notation of [GK]. In particular, if L is a closed subgroup of J, $\pi \in \mathrm{Irr}(L)$ and $g \in J$, then $g\pi \in \mathrm{Irr}(L)$ is the representation with the same space as π and action defined by $(g\pi)(h) = \pi(g^{-1}hg)$, and J_π is the subgroup of $g \in J$ such that $g\pi \cong \pi$. Throughout the paper k is a nonarchimedean local field of characteristic zero. Finally, let $(\ ,\ )_k$ denote the Hilbert symbol of k.

1. Howe duality and multiplicity preservation. For the convenience of the reader, we recall the statements of Howe duality and multiplicity preservation as in [R], [MVW] and [KR]. Let A and B be groups of td-type, with countable bases. Let $(\rho, \mathfrak{U})$ be a smooth representation of $A \times B$. Let $\pi \in \operatorname{Irr}(A)$. Define

$$\mathcal{U}(\pi) = \mathcal{U} / \bigcap_{t \in \operatorname{Hom}_A(\rho,\pi)} \ker(t).$$

Via ρ, $A \times B$ acts on $\mathcal{U}(\pi)$. Call this representation $\rho(\pi)$. By [MVW] there exists a smooth representation $\Theta(\pi)$ of B, unique up to isomorphism, such that

$$\rho(\pi) \cong \pi \otimes_{\mathbb{C}} \Theta(\pi)$$

as representations of $A \times B$. Analogous remarks apply for elements of $\operatorname{Irr}(B)$. Let $\mathcal{R}(A)$ be the set of equivalence classes of $\pi \in \operatorname{Irr}(A)$ such that $\mathcal{U}(\pi) \neq 0$. Define $\mathcal{R}(B)$ similarly. We say that **Strong Howe duality** holds for ρ if for every $\pi \in \mathcal{R}(A)$ the representation $\Theta(\pi)$ has a unique nonzero irreducible quotient $\theta(\pi) \in \mathcal{R}(B)$, and for every $\tau \in \mathcal{R}(B)$ the representation $\Theta(\tau)$ has a unique nonzero irreducible quotient $\theta(\tau) \in \mathcal{R}(A)$. We say that **Howe duality** holds for ρ if the set

$$\mathcal{R}(A \times B) = \{(\pi,\tau) \in \mathcal{R}(A) \times \mathcal{R}(B) \colon \operatorname{Hom}_{A\times B}(\rho, \pi \otimes_{\mathbb{C}} \tau) \neq 0\}$$

is the graph of a bijection between $\mathcal{R}(A)$ and $\mathcal{R}(B)$. Equivalently, Howe duality holds for ρ if and only if (1) every $\pi \in \mathcal{R}(A)$ occurs as the first entry of an element of $\mathcal{R}(A \times B)$ and every $\tau \in \mathcal{R}(B)$ occurs as the second entry of an element of $\mathcal{R}(A \times B)$; and (2) for all $\pi \in \operatorname{Irr}(A)$ and $\tau_1, \tau_2 \in \operatorname{Irr}(B)$,

$$\operatorname{Hom}_{A\times B}(\rho, \pi \otimes_{\mathbb{C}} \tau_1) \neq 0, \quad \operatorname{Hom}_{A\times B}(\rho, \pi \otimes_{\mathbb{C}} \tau_2) \neq 0 \implies \tau_1 \cong \tau_2;$$

for all $\pi_1, \pi_2 \in \operatorname{Irr}(A)$ and $\tau \in \operatorname{Irr}(B)$,

$$\operatorname{Hom}_{A\times B}(\rho, \pi_1 \otimes_{\mathbb{C}} \tau) \neq 0, \quad \operatorname{Hom}_{A\times B}(\rho, \pi_2 \otimes_{\mathbb{C}} \tau) \neq 0 \implies \pi_1 \cong \pi_2.$$

We say that **multiplicity preservation** holds for ρ if for all $\pi \in \operatorname{Irr}(A)$ and $\tau \in \operatorname{Irr}(B)$,

$$\dim_{\mathbb{C}} \operatorname{Hom}_{A\times B}(\rho, \pi \otimes_{\mathbb{C}} \tau) \leq 1.$$

The three properties are related by the following proposition:

Proposition 1.1. *Strong Howe duality holds for ρ if and only if Howe duality and multiplicity preservation hold for ρ. If strong Howe duality holds for ρ, then the map $\theta \colon \mathcal{R}(A) \to \mathcal{R}(B)$ is the bijection given by Howe duality.*

Proof. In the proof we will use the following fact. Let C and D be groups of td-type with countable bases. If $\pi \in \operatorname{Irr}(C)$ is nonzero and τ and τ' are representations of D, then $\operatorname{Hom}_C(\tau, \tau') \cong \operatorname{Hom}_{C\times D}(\pi \otimes_{\mathbb{C}} \tau, \pi \otimes_{\mathbb{C}} \tau')$ via the map that sends t to $1 \otimes t$.

Suppose that strong Howe duality holds for ρ. Let $\pi \in \mathcal{R}(A)$. The composition

$$\rho \to \rho(\pi) \to \pi \otimes_{\mathbb{C}} \Theta(\pi) \to \pi \otimes_{\mathbb{C}} \theta(\pi)$$

is a nonzero $A \times B$ map, and hence $(\pi, \theta(\pi)) \in \mathcal{R}(A \times B)$. Similarly, if $\tau \in \mathcal{R}(B)$, then $(\theta(\tau), \tau) \in \mathcal{R}(A \times B)$. This proves (1) of Howe duality for ρ. To prove (2), it suffices to show that if $\pi \in \mathrm{Irr}(A)$ and $\tau \in \mathrm{Irr}(B)$ and $\mathrm{Hom}_{A\times B}(\rho, \pi \otimes_{\mathbb{C}} \tau) \neq 0$, then $\tau \cong \theta(\pi)$ and $\pi \cong \theta(\tau)$. Fix a nonzero $A \times B$ map $\rho \to \pi \otimes_{\mathbb{C}} \tau$. Since $\rho(\pi) \cong \pi \otimes_{\mathbb{C}} \Theta(\pi)$, we have

$$\bigcap_{t \in \mathrm{Hom}_A(\rho,\pi)} \ker(t) = \bigcap_{U\ \mathbb{C}\ \text{vector space}} \ \bigcap_{t \in \mathrm{Hom}_A(\rho, \pi \otimes_{\mathbb{C}} U)} \ker(t).$$

It follows that our map factors through the canonical map $\rho \to \rho(\pi)$, so that there is a nonzero $A\times B$ map $\rho(\pi) \to \pi \otimes_{\mathbb{C}} \tau$. This implies that there is a nonzero B map $\Theta(\pi) \to \tau$. By strong Howe duality for ρ, $\tau \cong \theta(\pi)$. Similarly, $\pi \cong \theta(\tau)$. Note that we have also proven the second statement of the proposition.

Next, we prove that multiplicity preservation holds for ρ. Let $\pi \in \mathrm{Irr}(A)$ and $\tau \in \mathrm{Irr}(B)$. Let $T_1, T_2 \colon \rho \to \pi \otimes_{\mathbb{C}} \tau$ be $A \times B$ maps. We must show that T_1 and T_2 are linearly dependent. We may assume that one of the maps is nonzero. As a consequence of the last paragraph, we have $\tau \cong \theta(\pi)$. By the above characterization of

$$\bigcap_{t \in \mathrm{Hom}_A(\rho,\pi)} \ker(t)$$

it follows that $\ker(T_1)$ and $\ker(T_2)$ contain this set, so that we can regard T_1 and T_2 as maps from $\rho(\pi) \cong \pi \otimes_{\mathbb{C}} \Theta(\pi)$ to $\pi \otimes_{\mathbb{C}} \theta(\pi)$. Hence, $T_1 = 1 \otimes t_1$ and $T_2 = 1 \otimes t_2$ for some $t_1, t_2 \in \mathrm{Hom}_B(\Theta(\pi), \theta(\pi))$. By the uniqueness part of strong Howe duality for ρ and Schur's lemma, t_1 and t_2 are linearly dependent, and the same holds for T_1 and T_2.

Now suppose that Howe duality and multiplicity preservation hold for ρ. Let $\pi \in \mathcal{R}(A)$. By (1) of Howe duality for ρ, there exists $\tau \in \mathcal{R}(B)$ such that $\mathrm{Hom}_{A\times B}(\rho, \pi \otimes_{\mathbb{C}} \tau) \neq 0$. It follows that there is a nonzero $A \times B$ map $\rho(\pi) \cong \pi \otimes_{\mathbb{C}} \Theta(\pi) \to \pi \otimes_{\mathbb{C}} \tau$. Hence, there exists a nonzero B map $\Theta(\pi) \to \tau$. Thus, $\Theta(\pi)$ has a nonzero irreducible quotient. To see that this quotient is unique, suppose that σ_1 and σ_2 are subrepresentations of $\Theta(\pi)$ such that $\Theta(\pi)/\sigma_1$ and $\Theta(\pi)/\sigma_2$ are nonzero and irreducible. We must show that $\sigma_1 = \sigma_2$. Now there are nonzero $A \times B$ compositions:

$$\rho \to \rho(\pi) \to \pi \otimes_{\mathbb{C}} \Theta(\pi) \to \pi \otimes_{\mathbb{C}} \Theta(\pi)/\sigma_1,$$
$$\rho \to \rho(\pi) \to \pi \otimes_{\mathbb{C}} \Theta(\pi) \to \pi \otimes_{\mathbb{C}} \Theta(\pi)/\sigma_2.$$

By (2) of Howe duality, $\Theta(\pi)/\sigma_1 \cong \tau$ and $\Theta(\pi)/\sigma_2 \cong \tau$. Fix such isomorphisms. We then have nonzero $A \times B$ compositions:

$$\rho \to \rho(\pi) \to \pi \otimes_{\mathbb{C}} \Theta(\pi) \to \pi \otimes_{\mathbb{C}} \Theta(\pi)/\sigma_1 \to \pi \otimes_{\mathbb{C}} \tau,$$
$$\rho \to \rho(\pi) \to \pi \otimes_{\mathbb{C}} \Theta(\pi) \to \pi \otimes_{\mathbb{C}} \Theta(\pi)/\sigma_2 \to \pi \otimes_{\mathbb{C}} \tau.$$

By multiplicity preservation, these maps are nonzero multiples of each other. It follows that the compositions

$$\Theta(\pi) \to \Theta(\pi)/\sigma_1 \to \tau,$$
$$\Theta(\pi) \to \Theta(\pi)/\sigma_2 \to \tau$$

are nonzero multiples of each other. Hence, they have the same kernel, i.e., $\sigma_1 = \sigma_2$. □

We use the following proposition in section 6. Assume that A is contained in a group A' of td-type with countable basis as a closed normal subgroup of index two. Let a be a representative for the nontrivial coset of A'/A. Let $\rho' = \operatorname{Ind}_{A\times B}^{A'\times B} \rho$. All of the above definitions apply with ρ' in place of ρ. To avoid ambiguity, we will use subscripts to differentiate objects that could be defined with respect to both ρ and ρ'.

Proposition 1.2. *If Howe duality holds for ρ, then Howe duality holds for ρ' if and only if $\mathcal{R}(A) \cap a \cdot \mathcal{R}(A) = \emptyset$. If strong Howe duality holds for ρ, then strong Howe duality holds for ρ' if and only if $\mathcal{R}(A) \cap a \cdot \mathcal{R}(A) = \emptyset$.*

Proof. We first mention a fact that will be used several times in the proof. Let $\pi \in \operatorname{Irr}(A)$ and $\tau \in \operatorname{Irr}(B)$. Let $\pi' = \operatorname{Ind}_A^{A'} \pi$. If π^+ is an irreducible constituent of π', then

$$\operatorname{Hom}_{A\times B}(\rho, \pi \otimes_{\mathbb{C}} \tau) \neq 0 \implies \operatorname{Hom}_{A'\times B}(\rho', \pi^+ \otimes_{\mathbb{C}} \tau) \neq 0.$$

This follows from standard general properties of induced representations.

Suppose that Howe duality holds for ρ. Assume that Howe duality holds for ρ', and that $\mathcal{R}(A) \cap a \cdot \mathcal{R}(A) \neq 0$; we will obtain a contradiction. Let $\pi \in \mathcal{R}(A)$ be such that $a\pi \in \mathcal{R}(A)$. By (1) of Howe duality for ρ, there exist $\tau, \tau' \in \mathcal{R}_\rho(B)$ such that

$$\operatorname{Hom}_{A\times B}(\rho, \pi \otimes_{\mathbb{C}} \tau) \neq 0, \qquad \operatorname{Hom}_{A\times B}(\rho, a\pi \otimes_{\mathbb{C}} \tau') \neq 0.$$

Let $\pi' = \operatorname{Ind}_A^{A'} \pi = \operatorname{Ind}_A^{A'} a\pi$. First suppose that $\pi \ncong a\pi$. Then π' is irreducible, and

$$\operatorname{Hom}_{A'\times B}(\rho', \pi' \otimes_{\mathbb{C}} \tau) \neq 0, \qquad \operatorname{Hom}_{A'\times B}(\rho', \pi' \otimes_{\mathbb{C}} \tau') \neq 0.$$

By (2) of Howe duality for ρ', $\tau \cong \tau'$. This contradicts (2) of Howe duality for ρ. Suppose next that $\pi \cong a\pi$. Then $\pi' = \pi^+ \oplus \pi^-$ with $\pi^+, \pi^- \in \operatorname{Irr}(A')$, $\pi^+ \ncong \pi^-$, and

$$\operatorname{Hom}_{A'\times B}(\rho', \pi^+ \otimes_{\mathbb{C}} \tau) \neq 0, \qquad \operatorname{Hom}_{A'\times B}(\rho', \pi^- \otimes_{\mathbb{C}} \tau) \neq 0.$$

By (2) of Howe duality for ρ', $\pi^+ \cong \pi^-$, a contradiction.

Now suppose that $\mathcal{R}(A) \cap a \cdot \mathcal{R}(A) = \emptyset$. Let $\pi' \in \mathcal{R}(A')$, and let π be an irreducible constituent of $\pi'|_A$. Since $\rho'|_{A\times B} = \rho \oplus (a,1)\rho$, we may assume that $\pi \in \mathcal{R}(A)$. By $\mathcal{R}(A) \cap a \cdot \mathcal{R}(A) = \emptyset$ we have $\pi \ncong a\pi$, and so $\pi' = \operatorname{Ind}_A^{A'} \pi$. Also, by (1) of Howe duality for ρ, there exists $\tau \in \mathcal{R}_\rho(B)$ such that $\mathrm{Hom}_{A\times B}(\rho, \pi \otimes_{\mathbb{C}} \tau) \neq 0$. This implies that $\mathrm{Hom}_{A'\times B}(\rho', \pi' \otimes_{\mathbb{C}} \tau) \neq 0$, and proves half of (1) of Howe duality for ρ'. Suppose $\tau \in \mathcal{R}_{\rho'}(B)$. Then $\tau \in \mathcal{R}_\rho(B)$, and by (1) of Howe duality for ρ, there exists $\pi \in \mathcal{R}(A)$ such that $\mathrm{Hom}_{A\times B}(\rho, \pi\otimes_{\mathbb{C}}\tau) \neq 0$. Let $\pi' = \mathrm{Ind}_A^{A'} \pi$. Since $\mathcal{R}(A) \cap a \cdot \mathcal{R}(A) = \emptyset$, π' is irreducible, and so $\mathrm{Hom}_{A\times B}(\rho', \pi' \otimes_{\mathbb{C}} \tau) \neq 0$. This completes the proof of (1) of Howe duality for ρ'. To prove (2) of Howe duality for ρ', suppose that $\pi', \pi'' \in \mathrm{Irr}(A)$ and $\tau \in \mathrm{Irr}(B)$ are such that

$$\mathrm{Hom}_{A'\times B}(\rho', \pi' \otimes_{\mathbb{C}} \tau) \neq 0, \qquad \mathrm{Hom}_{A'\times B}(\rho', \pi'' \otimes_{\mathbb{C}} \tau) \neq 0.$$

Then $\pi', \pi'' \in \mathcal{R}(A')$, and as we have just seen, $\pi' = \mathrm{Ind}_A^{A'} \pi_1$ and $\pi'' = \mathrm{Ind}_A^{A'} \pi_2$ for some $\pi_1, \pi_2 \in \mathcal{R}(A)$. Since $\rho'|_{A\times B} = \rho \oplus (a,1)\rho$,

$$\mathrm{Hom}_{A\times B}(\rho, \pi_1 \otimes_{\mathbb{C}} \tau) \neq 0 \quad \text{or} \quad \mathrm{Hom}_{A\times B}(\rho, a\pi_1 \otimes_{\mathbb{C}} \tau) \neq 0,$$

and

$$\mathrm{Hom}_{A\times B}(\rho, \pi_2 \otimes_{\mathbb{C}} \tau) \neq 0, \quad \text{or} \quad \mathrm{Hom}_{A\times B}(\rho, a\pi_2 \otimes_{\mathbb{C}} \tau) \neq 0.$$

By (2) of Howe duality for ρ, $\pi_1 \cong \pi_2$ or $\pi_1 \cong a\pi_2$. This implies that $\pi \cong \pi'$. To complete the proof of (2) of Howe duality for ρ', let $\pi' \in \mathrm{Irr}(A')$ and $\tau, \tau' \in \mathrm{Irr}(B)$ be such that

$$\mathrm{Hom}_{A'\times B}(\rho', \pi' \otimes_{\mathbb{C}} \tau) \neq 0, \qquad \mathrm{Hom}_{A'\times B}(\rho', \pi' \otimes_{\mathbb{C}} \tau') \neq 0.$$

Again, $\pi' = \mathrm{Ind}_A^{A'} \pi$ for some $\pi \in \mathcal{R}(A)$, and

$$\mathrm{Hom}_{A\times B}(\rho, \pi \otimes_{\mathbb{C}} \tau) \neq 0 \quad \text{or} \quad \mathrm{Hom}_{A\times B}(\rho, a\pi \otimes_{\mathbb{C}} \tau) \neq 0,$$

and

$$\mathrm{Hom}_{A\times B}(\rho, \pi \otimes_{\mathbb{C}} \tau') \neq 0, \quad \text{or} \quad \mathrm{Hom}_{A\times B}(\rho, a\pi \otimes_{\mathbb{C}} \tau') \neq 0.$$

If the first and the fourth spaces were nonzero, or the second and the third spaces were nonzero, then $\mathcal{R}(A) \cap a \cdot \mathcal{R}(A) \neq \emptyset$, a contradiction. By (2) of Howe duality for ρ, the remaining possibilities imply that $\tau \cong \tau'$.

Finally, by Proposition 1.1 and the first statement of the proposition, to prove the second statement it suffices to show that for $\pi' \in \mathrm{Irr}(A')$ and $\tau \in \mathrm{Irr}(B)$,

$$\dim_{\mathbb{C}} \mathrm{Hom}_{A'\times B}(\rho', \pi' \otimes_{\mathbb{C}} \tau) \leq 1,$$

under the assumption of strong Howe duality for ρ, Howe duality for ρ', and $\mathcal{R}(A) \cap a \cdot \mathcal{R}(A) = \emptyset$. We may assume that $\mathrm{Hom}_{A'\times B}(\rho', \pi' \otimes_{\mathbb{C}} \tau) \neq 0$. As above, for some $\pi \in \mathcal{R}(A)$, we have $\pi' = \mathrm{Ind}_A^{A'} \pi$ and

$$\mathrm{Hom}_{A\times B}(\rho, \pi \otimes_{\mathbb{C}} \tau) \neq 0, \qquad \mathrm{Hom}_{A\times B}(\rho, a\pi \otimes_{\mathbb{C}} \tau) = 0.$$

Again by Proposition 1.2, to complete the proof it will suffice to show that the $\mathbb{C}$ linear map that takes an element $T \in \mathrm{Hom}_{A' \times B}(\rho', \pi' \otimes_{\mathbb{C}} \tau)$ to $T|_\rho$ composed with the projection $\pi' \otimes_{\mathbb{C}} \tau \to \pi \otimes_{\mathbb{C}} \tau$ is injective into $\mathrm{Hom}_{A \times B}(\rho, \pi \otimes_{\mathbb{C}} \tau)$. Here, we use the decompositions $\rho'|_{A \times B} = \rho \oplus (a, 1)\rho$ and $(\pi' \otimes_{\mathbb{C}} \tau)|_{A \times B} = (\pi \otimes_{\mathbb{C}} \tau) \oplus (a\pi \otimes_{\mathbb{C}} \tau)$. Suppose that T is in the kernel of this map. Then $T|_\rho = 0$, for otherwise $\mathrm{Hom}_{A \times B}(\rho, a\pi \otimes_{\mathbb{C}} \tau) \neq 0$. Since T is an $A' \times B$ map this implies $T|_{(a,1)\rho} = 0$. Thus, $T = 0$. □

Suppose that Howe duality holds for ρ and that $\mathcal{R}(A) \cap a \cdot \mathcal{R}(A) = \emptyset$, so that Howe duality holds for ρ'. Then the proof of Proposition 1.2 shows that $\mathcal{R}_{\rho'}(B) = \mathcal{R}_\rho(B)$, so the subscript is unnecessary. Also, if $\tau \in \mathcal{R}(B)$ then $\mathrm{Ind}_A^{A'} \theta_\rho(\tau)$ is irreducible and $\theta_{\rho'}(\tau) = \mathrm{Ind}_A^{A'} \theta_\rho(\tau)$. Finally, note that $\mathcal{R}(A) \cap a \cdot \mathcal{R}(A) = \emptyset$ implies $a \cdot \rho \ncong \rho$.

Given subsets $\mathrm{Irr}_0(A) \subset \mathrm{Irr}(A)$ and $\mathrm{Irr}_0(B) \subset \mathrm{Irr}(B)$, we can also define strong Howe duality, Howe duality and multiplicity preservation with respect to $\mathrm{Irr}_0(A)$ and $\mathrm{Irr}_0(B)$ by replacing $\mathrm{Irr}(A)$ and $\mathrm{Irr}(B)$ with $\mathrm{Irr}_0(A)$ and $\mathrm{Irr}_0(B)$, and $\mathcal{R}(A)$ and $\mathcal{R}(B)$ with $\mathcal{R}_0(A) = \mathcal{R}(A) \cap \mathrm{Irr}_0(A)$ and $\mathcal{R}_0(B) = \mathcal{R}(B) \cap \mathrm{Irr}_0(B)$, respectively. Proposition 1.1 remains true with the new definitions. Proposition 1.2 also remains true if $a \cdot \mathrm{Irr}_0(A) = \mathrm{Irr}_0(A)$ and $\mathrm{Irr}_0(A')$ is the set of $\pi \in \mathrm{Irr}(A')$ such that the constituents of $\pi|_A$ lie in $\mathrm{Irr}_0(A)$.

2. The groups. In this section we recall some facts about the metaplectic covers of similitude groups and the commutativity and splittings of inverse images from [B], [K] and [Ra]. Besides preparing for subsequent sections, our purpose is also to show that the situation for similitudes is not analogous to that for isometries.

Let $(\mathbb{W}, \langle\langle\ ,\ \rangle\rangle)$ be a finite dimensional nondegenerate symplectic vector space over k. Assume $\mathbb{W} \neq 0$. Let $\mathrm{GSp}(\mathbb{W})$ be the group of all k linear isomorphisms g from $\mathbb{W}$ to $\mathbb{W}$ such that there exists $\lambda \in k^\times$ such that $\langle\langle gw, gw' \rangle\rangle = \lambda \langle\langle w, w' \rangle\rangle$ for $w, w' \in \mathbb{W}$. If $g \in \mathrm{GSp}(\mathbb{W})$ then such a λ is unique, and will be denoted by $\lambda(g)$. Let $\mathrm{Sp}(\mathbb{W})$ be the subgroup of $g \in \mathrm{GSp}(\mathbb{W})$ such that $\lambda(g) = 1$. Note that we regard the elements of $\mathrm{GSp}(\mathbb{W})$ as acting on the left. Fix a complete polarization $\mathbb{W} = \mathbb{U} \oplus \mathbb{U}^*$ of $\mathbb{W}$. If necessary, we will write the elements of $\mathrm{GSp}(\mathbb{W})$ as matrices with respect to this polarization. Define an action of $k^\times$ on $\mathrm{Sp}(\mathbb{W})$ by

$$s^y = \begin{pmatrix} 1 & 0 \\ 0 & y \end{pmatrix}^{-1} s \begin{pmatrix} 1 & 0 \\ 0 & y \end{pmatrix}.$$

With respect to this action, $k^\times \ltimes \mathrm{Sp}(\mathbb{W}) \cong \mathrm{GSp}(\mathbb{W})$. The isomorphism is given by

$$(y, s) \mapsto \begin{pmatrix} 1 & 0 \\ 0 & y \end{pmatrix} s.$$

If $g \in \mathrm{GSp}(\mathbb{W})$, we let

$$g_1 = \begin{pmatrix} 1 & 0 \\ 0 & \lambda(g)^{-1} \end{pmatrix} g.$$

We recall the construction of the metaplectic cover of $\mathrm{GSp}(\mathbb{W})$ from [B]. Let $\mathbb{C}^1$ be the group of complex numbers of absolute value 1. To be consistent with [K] and [Ra] we will replace $\mathbb{C}^\times$ in [B] with $\mathbb{C}^1$. Let $\mathrm{Sp}(\mathbb{W})$ act trivially on $\{\pm 1\}$ and $\mathbb{C}^1$. It is known that $\mathrm{H}^2(\mathrm{Sp}(\mathbb{W}), \{\pm 1\})$ has order two. Fix a representative c for the image of the nontrivial class of $\mathrm{H}^2(\mathrm{Sp}(\mathbb{W}), \{\pm 1\})$ under the inclusion of $\mathrm{H}^2(\mathrm{Sp}(\mathbb{W}), \{\pm 1\})$ in $\mathrm{H}^2(\mathrm{Sp}(\mathbb{W}), \mathbb{C}^1)$. Let $\mathrm{Mp}(\mathbb{W})$ be the extension of $\mathrm{Sp}(\mathbb{W})$ by $\mathbb{C}^1$ defined by c. There is an exact sequence

$$1 \to \mathbb{C}^1 \to \mathrm{Mp}(\mathbb{W}) \to \mathrm{Sp}(\mathbb{W}) \to 1.$$

The typical element of $\mathrm{Mp}(\mathbb{W})$ will be denoted by (s, ϵ), where $s \in \mathrm{Sp}(\mathbb{W})$ and $\epsilon \in \mathbb{C}^1$. As in Lemma 1.1.B of [B], since $\mathrm{H}^2(\mathrm{Sp}(\mathbb{W}), \{\pm 1\})$ has order two, the action of each element of $k^\times$ on $\mathrm{Sp}(\mathbb{W})$ lifts uniquely to an action on $\mathrm{Mp}(\mathbb{W})$ that acts trivially on $\mathbb{C}^1$; since $\mathrm{Sp}(\mathbb{W})$ is perfect, for any extension of $\mathrm{Sp}(\mathbb{W})$ by $\mathbb{C}^1$ such a lifting is unique. This action again will be denoted by a superscript. It follows that there exists a function $v \colon k^\times \times \mathrm{Sp}(\mathbb{W}) \to \mathbb{C}^1$ such that

$$(s, \epsilon)^y = (s^y, v(y, s)\epsilon)$$

for $s \in \mathrm{Sp}(\mathbb{W})$, $\epsilon \in \mathbb{C}^1$, and $y \in k^\times$. Consider the semidirect product $k^\times \ltimes \mathrm{Mp}(\mathbb{W})$ corresponding to this action. Define a function $k^\times \ltimes \mathrm{Mp}(\mathbb{W}) \to \mathrm{GSp}(\mathbb{W}) \times \mathbb{C}^1$ by

$$(y, (s, \epsilon)) \to (\begin{pmatrix} 1 & 0 \\ 0 & y \end{pmatrix} s, \epsilon).$$

This function is a bijection. Via this bijection, the set $\mathrm{GSp}(\mathbb{W}) \times \mathbb{C}^1$ inherits a group structure from $k^\times \ltimes \mathrm{Mp}(\mathbb{W})$. A computation shows that the group law is given by

$$(g, \epsilon) \cdot (g', \epsilon') = (gg', C(g, g')\epsilon\epsilon'),$$

where $C \colon \mathrm{GSp}(\mathbb{W}) \times \mathrm{GSp}(\mathbb{W}) \to \mathbb{C}^1$ is defined by

$$C(g, g') = c(s^{y'}, s')v(y', s),$$

if (y, s) and (y', s') correspond to g and g', respectively. It follows that C is a cocycle. Denote the extension of $\mathrm{GSp}(\mathbb{W})$ by $\mathbb{C}^1$ corresponding to C by $\mathrm{GMp}(\mathbb{W})$. There is an exact sequence

$$1 \to \mathbb{C}^1 \to \mathrm{GMp}(\mathbb{W}) \xrightarrow{p} \mathrm{GSp}(\mathbb{W}) \to 1.$$

The typical element of $\mathrm{GMp}(\mathbb{W})$ will be denoted by (g, ϵ), where $g \in \mathrm{GSp}(\mathbb{W})$ and $\epsilon \in \mathbb{C}^1$. The equivalence class of $\mathrm{GMp}(\mathbb{W})$ does not depend on the choice of c or complete polarization. We call $\mathrm{GMp}(\mathbb{W})$ the **metaplectic cover** of $\mathrm{GSp}(\mathbb{W})$.

The preceding discussion remains valid if c is assumed to take values in $\{\pm 1\}$ and $\mathbb{C}^1$ is everywhere replaced by $\{\pm 1\}$. The result is a two-fold cover of $\mathrm{GSp}(\mathbb{W})$:

$$1 \to \{\pm 1\} \to \widehat{\mathrm{GSp}(\mathbb{W})} \to \mathrm{GSp}(\mathbb{W}) \to 1.$$

There is an inclusion of $\widehat{\mathrm{GSp}(\mathbb{W})}$ in $\mathrm{GMp}(\mathbb{W})$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{GMp}(\mathbb{W}) & \longrightarrow & \mathrm{GSp}(\mathbb{W}) \\ \uparrow & & \uparrow Id \\ \widehat{\mathrm{GSp}(\mathbb{W})} & \longrightarrow & \mathrm{GSp}(\mathbb{W}). \end{array}$$

Now let $(X, (\ ,\ ))$ be a nondegenerate symmetric bilinear space over k of dimension m, and let $(Y, \langle\ ,\ \rangle)$ be a nondegenerate symplectic bilinear space over k of dimension $2n$. For the remainder of the paper we will assume that

$$(\mathbb{W}, \langle\langle\ ,\ \rangle\rangle) = (X, (\ ,\ )) \otimes_k (Y, \langle\ ,\ \rangle),$$

and that there is a complete polarization $Y = U \oplus U^*$ such that $\mathbb{U} = X \otimes_k U$ and $\mathbb{U}^* = X \otimes_k U^*$. If necessary, we will write the elements of $\mathrm{GSp}(Y)$ as matrices with respect to the polarization $Y = U \oplus U^*$. Let $\mathrm{GO}(X)$ be the set of all k linear isomorphisms h from X to X such that there exists $\lambda \in k^\times$ such that $(hx, hx') = \lambda(x, x')$ for $x, x' \in X$. If $h \in \mathrm{GO}(X)$ then such a λ is unique, and will be denoted by $\lambda(h)$. Let $\mathrm{O}(X)$ be the subgroup of $h \in \mathrm{GO}(X)$ such that $\lambda(h) = 1$. There are inclusions

$$\mathrm{GO}(X) \hookrightarrow \mathrm{GSp}(\mathbb{W}), \qquad \mathrm{GSp}(Y) \hookrightarrow \mathrm{GSp}(\mathbb{W}).$$

We will investigate the preimages $p^{-1}(\mathrm{GO}(X))$ and $p^{-1}(\mathrm{GSp}(Y))$.

First we recall from Proposition 2.2.A of [B] that, in contrast to the case of isometries, the elements of $p^{-1}(\mathrm{GO}(X))$ and $p^{-1}(\mathrm{GSp}(Y))$ do not commute.

Next, we consider $p^{-1}(\mathrm{GO}(X))$ and $p^{-1}(\mathrm{GSp}(Y))$ as extensions of $\mathrm{GO}(X)$ and $\mathrm{GSp}(Y)$ by $\mathbb{C}^1$, respectively. The following proposition shows that, as extensions, $p^{-1}(\mathrm{GSp}(Y))$ and $p^{-1}(\mathrm{Sp}(Y))$ are analogous.

Proposition 2.1. *If m is even, then $p^{-1}(\mathrm{GSp}(Y))$ is trivial as an extension of $\mathrm{GSp}(Y)$ by $\mathbb{C}^1$. If m is odd, then $p^{-1}(\mathrm{GSp}(Y))$ is the metaplectic cover of $\mathrm{GSp}(Y)$.*

Proof. We need some notation. We will use notation and definitions from [Ra] and [K]. This requires a translation, since [Ra] and [K] regard elements of $\mathrm{Sp}(\mathbb{W})$ as acting on the right. We will implicitly make this translation. As in [B], let $e_1, \ldots, e_n$ and $f_1, \ldots, f_n$ be ordered bases for U and U^*, respectively, such that $e_1, \ldots, e_n, f_1, \ldots, f_n$ is a symplectic basis for Y. Let $v_1, \ldots, v_m$ be an orthogonal basis for X, and let $\alpha_1 = (v_1, v_1), \ldots, \alpha_m = (v_m, v_m)$. Then

$$v_1 \otimes e_1, \ldots, v_m \otimes e_n, \alpha_1^{-1} v_1 \otimes f_1, \ldots, \alpha_m^{-1} v_m \otimes f_n$$

is a symplectic basis for $\mathbb{W}$ such that the first mn vectors generate $\mathbb{U}$ and the second mn vectors generate $\mathbb{U}^*$. When we apply Proposition 1.2.A of [B] it will be with respect to this polarization and basis. Fix a nontrivial additive character ψ of k. We may assume that c is the unnormalized Rao cocycle corresponding to ψ, our polarization and basis. Let $\beta\colon \mathrm{Sp}(Y) \to \mathbb{C}^1$ be the function from Theorem 3.1 of [K] that is defined with respect to ψ and our polarization and basis of Y. The function β is used to compute the restriction of c to $\mathrm{Sp}(Y) \times \mathrm{Sp}(Y)$ regarded as a subgroup of $\mathrm{Sp}(\mathbb{W}) \times \mathrm{Sp}(\mathbb{W})$. For details, the reader should consult [K].

Suppose that m is even. Let $g, g' \in \mathrm{GSp}(Y)$. Then

$$C(1 \otimes g, 1 \otimes g') = c(1 \otimes g_1^{\lambda(g')}, 1 \otimes g_1')v(\lambda(g'), 1 \otimes g_1).$$

By Theorem 3.1 of [K],

$$c(1 \otimes g_1^{\lambda(g')}, 1 \otimes g_1') = \beta(g_1^{\lambda(g')})^{-1}\beta(g_1')^{-1}\beta((gg')_1).$$

By the definition of β, Proposition 1.2.A of [B], and Corollary A.5 of [Ra],

$$v(\lambda(g'), 1 \otimes g_1)\beta(g_1^{\lambda(g')})^{-1} = \beta(g_1)^{-1}.$$

It follows that

$$C(1 \otimes g, 1 \otimes g') = \beta(g_1)^{-1}\beta(g_1')^{-1}\beta((gg')_1),$$

and thus $p^{-1}(\mathrm{GSp}(Y))$ is trivial.

Suppose m is odd. Let $g, g' \in \mathrm{GSp}(Y)$. Then by Theorem 3.1 of [K],

$$C(1 \otimes g, 1 \otimes g') = c'(g_1^{\lambda(g')}, g_1')w(\lambda(g'), g_1),$$

where $c'\colon \mathrm{Sp}(Y) \times \mathrm{Sp}(Y) \to \mathbb{C}^1$ is defined by

$$c'(s, s') = \beta(s)^{-1}\beta(s')^{-1}\beta(ss')c^0(s, s'),$$

and $w\colon k^\times \times \mathrm{Sp}(Y) \to \mathbb{C}^1$ is defined by

$$w(y, s) = v(y, 1 \otimes s).$$

Here c^0 is the normalization of c, as in section 5 of [Ra]. The cocycle c' is nontrivial. Hence, to complete the proof it suffices to show that $w = v'$, where $v'\colon k^\times \times \mathrm{Sp}(Y) \to \mathbb{C}^1$ is the function associated to c' as above. By section 5 of [Ra],

$$c'(s, s') = d(s)d(s')d(ss')^{-1}c_U(s, s'),$$

for $y \in \mathrm{Sp}(Y)$, where $d\colon \mathrm{Sp}(Y) \to \mathbb{C}^1$ is defined by

$$d(s) = \frac{m(s)}{\beta(s)}$$

and $m\colon \mathrm{Sp}(Y) \to \mathbb{C}^1$ is as in section 5 of [Ra]. Here c_U is the unnormalized Rao cocycle corresponding to U and ψ. By the uniqueness of v',

$$v'(y,s) = \frac{d(s)}{d(s^y)} v_U(y,s).$$

for $y \in k^\times$ and $s \in \mathrm{Sp}(Y)$. It thus suffices to show that

$$\frac{w(y,s)}{v_U(y,s)} = \frac{d(s)}{d(s^y)}$$

for $y \in k^\times$ and $s \in \mathrm{Sp}(Y)$. This follows by a computation using Proposition 1.2.A of [B]. □

Lemma 2.2. *Assume that the residual characteristic of k is odd. Suppose that m is even, and $m \geq 4$. If X is not the four dimensional anisotropic symmetric bilinear space, then* $[\mathrm{SO}(X), \mathrm{SO}(X)] = [\mathrm{GO}(X), \mathrm{GO}(X)]$.

Proof. Let $\mathrm{SO}(X)'$ be the kernel of the spinor norm θ restricted to $\mathrm{SO}(X)$, as in section 55 of [O]. By 43:7 and 95:1 of [O], $[\mathrm{SO}(X), \mathrm{SO}(X)] = \mathrm{SO}(X)'$. It thus suffices to show that $[\mathrm{GO}(X), \mathrm{GO}(X)] \subset \mathrm{SO}(X)'$. Note first that if $h \in \mathrm{GO}(X)$ and $h_1 \in \mathrm{O}(X)$ then $\theta(hh_1h^{-1}) = \lambda(h)\theta(h_1)$. Since the residual characteristic of k is odd, there exist coset representatives for $\mathrm{GO}(X)/k^\times \mathrm{O}(X)$ that commute. This can be shown using the Witt decomposition of X and the standard models for the groups of similitudes of two and four dimensional nondegenerate symmetric bilinear spaces. See [La] and [HPS] for these models. Let $h, h' \in \mathrm{GO}(X)$. There exist $h_0, h_0' \in \mathrm{GO}(X)$, $a, a' \in k^\times$ and $h_1, h_1' \in \mathrm{O}(X)$ such that $h = ah_0h_1$, $h' = a'h_0'h_1'$ and h_0 and h_0' commute. Now

$$[h,h'] = h_0h_0'({h_0'}^{-1}h_1h_0')h_1'h_1^{-1}(h_0^{-1}{h_1'}^{-1}h_0)(h_0h_0')^{-1}.$$

It follows that $\theta([h,h']) = 1$. □

The next proposition shows that, as extensions, $p^{-1}(\mathrm{GO}(X))$ and $p^{-1}(\mathrm{O}(X))$ are not analogous.

Proposition 2.3. *Assume that the residual characteristic of k is odd. If m is odd, $m = 2$, or $m = 4$ and X is anisotropic, then $p^{-1}(\mathrm{GO}(X))$ is trivial as an extension of $\mathrm{GO}(X)$ by $\mathbb{C}^1$. If m is even, $m \geq 4$, and X is not four dimensional and anisotropic, then $p^{-1}(\mathrm{GO}(X))$ is trivial if and only if the character of $\mathrm{GSO}(X)$ defined by $h \mapsto (-1, \lambda(h))_k^n$ is trivial.*

Proof. We may assume that c is as in the proof of Proposition 2.1. Let $h, h' \in \mathrm{GO}(X)$. By Proposition 1.2.A of [B],

$$C(h \otimes 1, h' \otimes 1) = (\det(h), \lambda(h'))_k^n.$$

If m is odd then the statement follows from the identity

$$(\det(h)/\lambda(h)^{(m-1)/2})^2 = \lambda(h)$$

for $h \in \mathrm{GO}(X)$.

Suppose m is even. The following argument was suggested to me by S.S. Kudla. As usual, let $\mathrm{GSO}(X)$ be the kernel of the homomorphism from $\mathrm{GO}(X)$ to $k^\times$ defined by $h \mapsto \det(h)/\lambda(h)^{m/2}$. Let $\tau \in \mathrm{O}(X)$ be a symmetry as in 42E of [O]. Then $\mathrm{GO}(X) = \mathrm{GSO}(X) \rtimes \langle \tau \rangle$. Define $\gamma\colon \mathrm{GSO}(X) \to \mathbb{C}^1$ by $\gamma(h) = \gamma_k(\lambda(h), \psi)^{mn/2}$. Here γ_k is the Weil index as in the Appendix of [Ra]. Then by Theorem A.4 of [Ra],

$$C(h \otimes 1, h' \otimes 1) = \gamma(h)^{-1}\gamma(h')^{-1}\gamma(hh'),$$

for $h, h' \in \mathrm{GSO}(X)$. Let χ be a unitary character of $\mathrm{GSO}(X)$. Define $i\colon \mathrm{GSO}(X) \to p^{-1}(\mathrm{GO}(X))$ by $i(h) = (h \otimes 1, \chi(h)\gamma(h))$. Then i is a homomorphism, and any homomorphism i' from $\mathrm{GSO}(X)$ to $p^{-1}(\mathrm{GO}(X))$ such that $p(i'(h)) = h \otimes 1$ is of this form. It follows that $p^{-1}(\mathrm{GO}(X))$ is trivial if and only if there exists a χ and $\epsilon \in \{\pm 1\}$ such that

$$i(\tau h \tau^{-1}) = (\tau \otimes 1, \epsilon) i(h) (\tau \otimes 1, \epsilon)^{-1}.$$

A computation shows that this is equivalent to the existence of a χ such that

$$\chi([\tau, h]) = (-1, \lambda(h))_k^n$$

for $h \in \mathrm{GSO}(X)$. By Lemma 2.2 it now suffices to consider the case $m = 2$ and the case $m = 4$ and X is anisotropic. In these cases, using the standard models mentioned in the proof of Lemma 2.2, one can show that such a χ exists. $\square$

3. The representations. Let $(r, \mathcal{S})$ be a model of the smooth Weil representation of $\mathrm{Mp}(\mathbb{W})$ corresponding to ψ. In this section we give a unified account of the two approaches mentioned in the introduction to constructing an analogue for similitudes of the restriction of r to $p^{-1}(\mathrm{O}(X))p^{-1}(\mathrm{Sp}(X))$. Our main point is that both constructions, as well as their relationship to each other, depend on the fundamental identity of Lemma 3.2. To avoid groups that are not of td-type, and to make applications easier, we will introduce splittings whenever possible. The development does not depend on these splittings, and one could avoid their introduction by proceeding as in [MVW] or [B]. In this section c is as in the last section.

To make the key observation, suppose for the moment that c is the Rao cocycle from the proof of Proposition 2.1 and $(r, \mathcal{S})$ is the Scrödinger model corresponding to the polarization $\mathbb{W} = \mathbb{U} \oplus \mathbb{U}^*$ as in [B]. Then $\mathrm{GO}(X)$ acts naturally on $\mathcal{S}(\mathbb{U}^*)$ via left translation, and

$$r(\begin{pmatrix} \lambda(h)^{-1} & 0 \\ 0 & 1 \end{pmatrix}(h \otimes 1), \epsilon)\varphi(x) = \epsilon |\lambda(h)|^{-\frac{mn}{4}} \varphi((h^{-1} \otimes 1)x)$$

for $\varphi \in \mathcal{S} = \mathcal{S}(\mathbb{U}^*)$, $h \in \mathrm{GO}(X)$ and $\epsilon \in \mathbb{C}^1$. This suggests that instead of the natural image of $\mathrm{GO}(X)$ in $\mathrm{GSp}(\mathbb{W})$ we should consider the isomorphic subgroup generated by the elements

$$\begin{pmatrix} \lambda(h)^{-1} & 0 \\ 0 & 1 \end{pmatrix} (h \otimes 1)$$

for $h \in \mathrm{GO}(X)$.

The next proposition shows that, as extensions, the preimage of this subgroup and $p^{-1}(\mathrm{O}(X))$ are analogous.

Proposition 3.1. *The preimage under p of the above subgroup is trivial as an extension of $\mathrm{GO}(X)$ by $\mathbb{C}^1$.*

Proof. We may assume that c is as in the proof of Proposition 2.1. A computation shows that the restriction of C to the preimage is 1. □

Fix a splitting of the preimage. That is, fix a monomorphism $L\colon \mathrm{GO}(X) \hookrightarrow \mathrm{Mp}(\mathbb{W})$ of the form

$$L(h) = (\begin{pmatrix} \lambda(h)^{-1} & 0 \\ 0 & 1 \end{pmatrix} (h \otimes 1), \delta(h))$$

for $h \in \mathrm{GO}(X)$. If $h \in \mathrm{O}(X)$, we will write h for $L(h)$.

The following lemma is the analogue of the commutativity of the inverse images in the case of isometries.

Lemma 3.2 (Fundamental identity). *If $h \in \mathrm{GO}(X)$ and $g \in p^{-1}(\mathrm{Sp}(Y))$ then*

$$L(h)gL(h)^{-1} = g^{\lambda(h)^{-1}}.$$

Proof. Now $L(h)p^{-1}(\mathrm{Sp}(Y))L(h)^{-1} = p^{-1}(\mathrm{Sp}(Y))$, so that $g \mapsto L(h)gL(h)^{-1}$ defines an automorphism of $p^{-1}(\mathrm{Sp}(Y))$. Moreover, this automorphism fixes $\mathbb{C}^1$ pointwise, and $p(L(h)gL(h)^{-1}) = p(g)^{\lambda(h)^{-1}}$ for $g \in p^{-1}(\mathrm{Sp}(Y))$. Since $\mathrm{Sp}(Y)$ is perfect there can be at most one such automorphism, and the lemma follows. □

If m is even, let $G' = \mathrm{GSp}(Y)$. If m is odd, let G' be the two-fold cover of $\mathrm{GSp}(Y)$ discussed in the last section, and for $g \in G'$, let $\lambda(g)$ be the similitude factor of the projection of g to $\mathrm{GSp}(Y)$. Let G_1 be the subgroup of $g \in G'$ such that $\lambda(g) = 1$. Let $y \in k^\times$; if m is even, let $d(y) \in G'$ be

$$\begin{pmatrix} 1 & 0 \\ 0 & y \end{pmatrix};$$

if m is odd, let $d(y) \in G'$ be

$$(\begin{pmatrix} 1 & 0 \\ 0 & y \end{pmatrix}, 1).$$

The map $y \mapsto d(y)$ is a homomorphism. If m is even, then by Proposition 2.1 $p^{-1}(\mathrm{GSp}(Y))$ is trivial as an extension of $\mathrm{GSp}(Y)$ by $\mathbb{C}^1$. In this case, fix a splitting $G' \hookrightarrow p^{-1}(\mathrm{GSp}(Y))$. If m is odd, then by Proposition 2.1, $p^{-1}(\mathrm{GSp}(Y))$ is the metaplectic cover of $\mathrm{GSp}(Y)$. In this case, fix an inclusion $G' \hookrightarrow p^{-1}(\mathrm{GSp}(Y))$ as discussed in the last section. In what follows, we will implicitly use the inclusion of G' in $\mathrm{GMp}(\mathbb{W})$. Note that by an argument similar to the proof of Proposition 3.2, if $y \in k^\times$ and $g \in \mathrm{GMp}(\mathbb{W})$ then

$$d(y)^{-1} g d(y) = g^y.$$

Let $H_1 = \mathrm{O}(X)$ and $H = \mathrm{GO}(X)$.

When m is even, the following proposition appears in [HK]. It is implicit in [S].

Proposition 3.3 (Shimizu-Harris-Kudla). *Define an action of H on G_1 by $h \cdot g = g^{\lambda(h)^{-1}}$ and form the semidirect product $G_1 \rtimes H$. The map $G_1 \rtimes H \to \mathrm{Mp}(\mathbb{W})$ defined by $(g, h) \mapsto gL(h)$ is a homomorphism. Let*

$$R = \{(g, h) \in G' \times H \colon \lambda(g) = \lambda(h)\}.$$

The map $R \to G_1 \rtimes H$ defined by

$$(g, h) \mapsto (gd(\lambda(g))^{-1}, h)$$

is a homomorphism. Thus, the composition ω

$$R \to G_1 \rtimes H \to \mathrm{Mp}(\mathbb{W}) \xrightarrow{r} Aut_{\mathbb{C}}(\mathcal{S})$$

is a homomorphism. This representation is smooth.

Proof. This follows by the fundamental identity and standard arguments. □

We call ω the **extended Weil representation** associated to X and Y. The name is a consequence of the important fact that

$$\omega(g, h) = r(gh)$$

for $g \in G_1$ and $h \in H_1$.

The proof of the next proposition is similar. We are not certain to whom this proposition should be credited.

Proposition 3.4. *Let $\Omega = \text{c-Ind}_{G_1}^{G'} r$. For each $h \in H$, define an operator $\Omega(h)$ on the space $\mathcal{T}$ of Ω by*

$$(\Omega(h)f)(g) = r(L(h)) \cdot f(d(\lambda(h))^{-1} g).$$

Then the map

$$\Omega \colon G' \times H \to Aut_{\mathbb{C}}(\mathcal{T})$$

defined by $(g,h) \mapsto \Omega(g)\Omega(h)$ is a homomorphism. This representation is smooth.

We call Ω the **induced Weil representation** associated to X and Y. To see that $(\Omega, \mathcal{T})$ is the same representation as in the papers mentioned in the introduction, suppose again that c is the Rao cocycle from the proof of Proposition 2.1 and $(r, \mathcal{S})$ is the above Schrödinger model. Define $\mathcal{T} \to \mathcal{S}(\mathbb{U}^* \times k^\times)$ by $f \mapsto \phi$, where

$$\phi(x,y) = f(d(y)^{-1})(x).$$

This map is an isomorphism of $\mathbb{C}$-vector spaces. The resulting action on $\mathcal{S}(\mathbb{U}^* \times k^\times)$ is:

$$\begin{aligned} \Omega(1,h)\phi(x,y) &= |\lambda(h)|^{-\frac{nm}{4}}\phi((h^{-1}\otimes 1)x, \lambda(h)y), \\ \Omega(s,1)\phi(x,y) &= r(s^y)(\phi_y)(x), \\ \Omega(d(a),1)\phi(x,y) &= \phi(x, a^{-1}y), \end{aligned}$$

where $h \in H$, $s \in G_1$, $a \in k^\times$, and $\phi_y(x) = \phi(x,y)$.

The next proposition, which seems to be new, relates the extended and induced Weil representations.

Proposition 3.5. *We have*

$$\Omega \cong \text{c-Ind}_R^{G'\times H}\,\omega.$$

Proof. Define $T\colon \Omega \to \text{c-Ind}_R^{G'\times H}\,\omega$ by $T(f)(g,h) = (\Omega(g,h)f)(1)$. Again, standard arguments and the fundamental identity show that T is a well defined isomorphism. □

In the next section we prove that the analogue of Howe duality and multiplicity preservation hold for ω, and in subsequent sections we investigate, via Frobenius reciprocity, the consequences for Ω. Since in some cases R fails to involve all of G', we require an intermediate group and representation.

Let G be the group of $g \in G'$ such that $\lambda(g) \in \lambda(H)$. Then $G = G'$ except if m is even and the Witt index of X is $(m-2)/2$, and if m is odd. In the first case, X has Witt decomposition:

$$X \cong K \perp \mathbb{H} \perp \cdots \perp \mathbb{H}.$$

Here $\mathbb{H}$ is the hyperbolic plane over k, K is a quadratic extension of k, and the symmetric bilinear form on K is given by $(x,y) \mapsto a\,\mathrm{T}_k^K(x\overline{y})$, where $-$ is the nontrivial element of $\mathrm{Gal}(K/k)$, and $a \in k^\times$. By the uniqueness of the Witt decomposition, $\lambda(H) = \mathrm{N}_k^K(K^\times)$, so that $[G' : G] = [k^\times : \mathrm{N}_k^K(K^\times)] = 2$. In the second case, $\lambda(H) = k^{\times 2}$. Thus, in this case, $[G' : G] = [k^\times : k^{\times 2}] > 1$. Evidently, G is a closed subgroup of G'.

Define a smooth subrepresentation $(\Omega^+, \mathcal{T}^+)$ of $\Omega|_G$ in the following way. Let $\mathcal{T}^+$ be the $\mathbb{C}$ subspace of $f \in \mathcal{T}$ supported in G. The restriction of Ω to $G \times H$ acts on $\mathcal{T}^+$, and we call the resulting representation Ω^+. As above, there is an isomorphism

$$\Omega^+ \cong \text{c-Ind}_R^{G\times H}\,\omega.$$

In the next three sections, we show that strong Howe duality holds for Ω^+ with respect to certain natural subsets of $\mathrm{Irr}(G)$ and $\mathrm{Irr}(H)$, while Ω does not always satisfy strong Howe duality.

In the next section we will use the following fact: there exist closed subgroups $Z \subset \mathrm{Z}(G)$ and $Z' \subset \mathrm{Z}(H)$ and an isomorphism $\iota\colon Z \to Z'$ such that G/ZG_1 and $H/Z'H_1$ are finite abelian groups, and $\omega(z, \iota(z)) = \chi(z)$ for a character χ of Z.

4. Howe duality and multiplicity preservation for ω. Consider $\omega|_{G_1 \times H_1}$. As a consequence of a more general theorem of J.-L. Waldspurger that appears in [W] we have:

Theorem 4.1 (Waldspurger). *If the residual characteristic of k is odd then strong Howe duality holds for $\omega|_{G_1 \times H_1}$.*

Let $\mathrm{Irr}_0(G)$ be the set of $\pi \in \mathrm{Irr}(G)$ such that $\pi|_{G_1}$ is multiplicity free, and let $\mathcal{R}_0(G)$ be the set of $\pi \in \mathrm{Irr}_0(G)$ such that some constituent of $\pi|_{G_1}$ lies in $\mathcal{R}(G_1)$. Define $\mathrm{Irr}_0(H)$ and $\mathcal{R}_0(H)$ similarly. Here $\mathcal{R}(G_1)$ and $\mathcal{R}(H_1)$ are defined as in section 1, with respect to $\omega|_{G_1 \times H_1}$. In this section we will prove that the condition

$$\mathrm{Hom}_R(\omega, \pi \otimes_{\mathbb{C}} \tau) \neq 0$$

for $\pi \in \mathrm{Irr}_0(G)$ and $\tau \in \mathrm{Irr}_0(H)$ defines a bijection between $\mathcal{R}_0(G)$ and $\mathcal{R}_0(H)$, and that

$$\dim_{\mathbb{C}} \mathrm{Hom}_R(\omega, \pi \otimes_{\mathbb{C}} \tau) \leq 1$$

for $\pi \in \mathrm{Irr}_0(G)$ and $\tau \in \mathrm{Irr}_0(H)$. Thus, we will show that the analogues of (1) and (2) of Howe duality and multiplicity preservation hold for ω.

As we shall see, these results depend only on general properties of ω, and Theorem 4.1. Thus, in this section we work with an abstraction of ω that satisfies the conclusion of Theorem 4.1. Perhaps the theory will apply to other situations. Our main tool is the restriction theory of [GK].

In this section we will assume the following. Let G and H be groups of td-type with countable bases. Let G_1 and H_1 be closed normal subgroups of G and H respectively, and let Z and Z' be closed subgroups of $\mathrm{Z}(G)$ and $\mathrm{Z}(H)$, respectively. We assume that ZG_1 and $Z'H_1$ are closed and that G/ZG_1 and $H/Z'H_1$ are finite abelian groups. It follows that ZG_1 and $Z'H_1$ are open. We assume that the maps of $Z \times G_1$ to ZG_1 and of $Z' \times H_1$ to $Z'H_1$ are open. This implies that if $\pi \in \mathrm{Irr}(ZG_1)$ and $\tau \in \mathrm{Irr}(Z'H_1)$ then $\pi|_{G_1} \in \mathrm{Irr}(G_1)$ and $\tau|_{H_1} \in \mathrm{Irr}(H_1)$. We will often regard elements of $\mathrm{Irr}(ZG_1)$ and $\mathrm{Irr}(Z'H_1)$ as elements of $\mathrm{Irr}(G_1)$ and $\mathrm{Irr}(H_1)$, respectively. Let R be a closed normal subgroup of $G \times H$ such that for every $g \in G$ there exists $h \in H$ such that $(g, h) \in R$, and that for every $h \in H$ there exists $g \in G$ such that $(g, h) \in R$. Assume that

$$G_1 \times H_1 \subset R, \qquad (G \times 1) \cap R = G_1 \times 1, \qquad (1 \times H) \cap R = 1 \times H_1.$$

Assume that there is an isomorphism $\iota\colon Z \to Z'$ such that $\{(z, \iota(z))\colon z \in Z\} \subset R$. Then $(G_1 \times H_1)\{(z, \iota(z))\colon z \in Z\}$ is closed in R, and there are isomorphisms

$$G/ZG_1 \cong R/(G_1 \times H_1)\{(z, \iota(z))\colon z \in Z\} \cong H/Z'H_1.$$

Let $(\omega, \mathcal{S})$ be a smooth representation of R. Finally, assume that there exists a character χ of Z such that $\omega(z, \iota(z)) = \chi(z)$ for $z \in Z$, and that strong Howe duality holds for $\omega|_{G_1 \times H_1}$.

We have the following basic lemmas.

Lemma 4.2. *Let $\pi \in \mathrm{Irr}(G)$ and $\tau \in \mathrm{Irr}(H)$. Suppose that*

$$\mathrm{Hom}_R(\omega, \pi \otimes_{\mathbb{C}} \tau) \neq 0.$$

Then

$$\pi|_{G_1} \cong m \cdot \pi_1 \oplus \cdots \oplus m \cdot \pi_M, \qquad \tau|_{H_1} \cong m' \cdot \theta(\pi_1) \oplus \cdots \oplus m' \cdot \theta(\pi_M),$$

where the π_i are pairwise inequivalent elements of $\mathcal{R}(G_1)$, and m and m' are positive integers. Moreover, $m = 1$ if and only if $m' = 1$.

Proof. By Lemma 2.1 of [GK],

$$\pi|_{ZG_1} \cong m \cdot \pi_1 \oplus \cdots \oplus m \cdot \pi_M, \qquad \tau|_{Z'H_1} \cong m' \cdot \tau_1 \oplus \cdots \oplus m' \cdot \tau_{M'}$$

where the $\pi_i \in \mathrm{Irr}(ZG_1)$ are pairwise inequivalent, the $\tau_j \in \mathrm{Irr}(Z'H_1)$ are pairwise inequivalent, and m, m', M, and M' are positive integers. It follow that the π_i are pairwise inequivalent elements of $\mathrm{Irr}(G_1)$ and the τ_j are pairwise inequivalent elements of $\mathrm{Irr}(H_1)$. Let V and W be the spaces of π and τ, respectively, and let V_i and W_j be the spaces of $m \cdot \pi_i$ and $m' \cdot \tau_j$ respectively, so that

$$V = V_1 \oplus \cdots \oplus V_M, \qquad W = W_1 \oplus \cdots \oplus W_{M'}.$$

By the proof of Lemma 2.1 of [GK], G and H act transitively on the V_i and W_j, respectively.

Now let $\omega \to \pi \otimes_{\mathbb{C}} \tau$ be a nonzero R map. Since this map is nonzero, we may assume, after renumbering, that the composition

$$\mathcal{S} \to V \otimes_{\mathbb{C}} W \to V_1 \otimes_{\mathbb{C}} W_1$$

is a nonzero $G_1 \times H_1$ map. Here the second map is projection. Fix i between 1 and M. Let $g \in G$ be such that $\pi(g)V_1 = V_i$. Fix $h \in H$ such that $(g, h) \in R$. There exists $j(i)$ between 1 and M' such that $\tau(h)W_1 = W_{j(i)}$. Consider the nonzero composition

$$\mathcal{S} \to \mathcal{S} \to V \otimes_{\mathbb{C}} W \to V_1 \otimes_{\mathbb{C}} W_1 \to V_i \otimes_{\mathbb{C}} W_{j(i)}.$$

Here the first map is given by $\varphi \mapsto \omega(g^{-1}, h^{-1})\varphi$ and the last map is $\pi(g) \otimes \tau(h)$. A computation shows that this map is the $G_1 \times H_1$ map

$$\mathcal{S} \to V \otimes_{\mathbb{C}} W \to V_i \otimes_{\mathbb{C}} W_{j(i)},$$

where the last map is projection. By composing with another projection, we obtain a nonzero $G_1 \times H_1$ map

$$\omega \to \pi_i \otimes_{\mathbb{C}} \tau_{j(i)}.$$

Hence, $\tau_{j(i)} \cong \theta(\pi_i)$. We have established the existence of a map $i \mapsto j(i)$ so that $\tau_{j(i)} \cong \theta(\pi_i)$. Similarly, there exists a map $j \mapsto i(j)$ so that $\pi_{i(j)} \cong \theta(\tau_j)$. Clearly, these maps are inverses of each other. The first assertion of the lemma now follows.

Next, we show that the image U of the nonzero $G_1 \times H_1$ map

$$\mathcal{S} \to V \otimes_{\mathbb{C}} W \to V_1 \otimes_{\mathbb{C}} W_1$$

is isomorphic to $\pi_1 \otimes_{\mathbb{C}} \theta(\pi_1)$ as a $G_1 \times H_1$ representation. Since V_1 is the direct sum of copies of π_1 as a G_1 representation,

$$\bigcap_{t \in \mathrm{Hom}_{G_1}(\mathcal{S}, V_1)} \ker(t) = \bigcap_{t \in \mathrm{Hom}_{G_1}(\omega, \pi_1)} \ker(t).$$

This implies that the above map factors through the map

$$\omega \to \omega(\pi_1) \cong \pi_1 \otimes_{\mathbb{C}} \Theta(\pi_1),$$

which in turn implies that our map factors through the map

$$\omega \to \pi_1 \otimes_{\mathbb{C}} \theta(\pi_1).$$

This proves our claim.

Suppose now that $m = 1$. By the lemma on page 45 of [MVW], there exists an H_1 subspace W' of W_1 such that $U = V_1 \otimes_{\mathbb{C}} W'$. By the result of the last paragraph, $W' \cong \theta(\pi_1)$ as an H_1 space, and so W' is irreducible as an H_1 space. Suppose that $m' > 1$. Then there exists $h \in H$ such that $\tau(h)W_1 = W_1$ and $\tau(h)W' \cap W' = 0$. Let $g \in G$ be such that $(g, h) \in R$. Evidently, $\pi(g)V_1 = V_1$. By the definition of U, since $\pi(g)V_1 = V_1$ and $\tau(h)W_1 = W_1$, U is invariant under $\pi(g) \otimes \tau(h)$. That is, $V_1 \otimes_{\mathbb{C}} \tau(h)W' = V_1 \otimes_{\mathbb{C}} W'$. This contradicts $\tau(h)W' \cap W' = 0$. Hence $m' = 1$. The converse has an entirely analogous proof. □

For the notation G_{π_1} and H_{τ_1} in the statement of the following lemma, see the end of the introduction to the paper.

Lemma 4.3. *Let $\pi_1 \in \mathrm{Irr}(ZG_1)$, and assume that π_1 has an extension $\widetilde{\pi_1} \in \mathrm{Irr}(G_{\pi_1})$. Let $\tau_1 \in \mathrm{Irr}(Z'H_1)$, and assume that there exists a nonzero $R \cap (ZG_1 \times Z'H_1)$ map*

$$T\colon \omega \to \pi_1 \otimes_{\mathbb{C}} \tau_1.$$

Then there exists an extension $\widetilde{\tau_1} \in \mathrm{Irr}(H_{\tau_1})$ of τ_1 to H_{τ_1} such that T is a $R \cap (G_{\pi_1} \times H_{\tau_1})$ map from ω to $\widetilde{\pi_1} \otimes_{\mathbb{C}} \widetilde{\tau_1}$. Moreover, the analogous results hold if the roles of ZG_1 and $Z'H_1$ are interchanged.

Proof. We show first that it suffices to prove that there exist some extensions $\widehat{\pi_1} \in \mathrm{Irr}(G_{\pi_1})$ and $\widehat{\tau_1} \in \mathrm{Irr}(H_{\tau_1})$ of π_1 and τ_1, respectively, such that T is an $R \cap (G_{\pi_1} \times H_{\tau_1})$ map. For suppose such extensions exist. By Lemma 2.4 of [GK] there exists a character v of G_{π_1} such that $\widehat{\pi_1} = v \otimes_{\mathbb{C}} \widetilde{\pi_1}$ and $v(ZG_1) = 1$. Define a function

$$H_{\tau_1}/Z'H_1 \to G_{\pi_1}/ZG_1$$

by $hZ'H_1 \mapsto gZG_1$, where $g \in G$ is such that $(g,h) \in R$. To show that this map is a well defined isomorphism it suffices to show that if $(g,h) \in R$ and $g \in G_{\pi_1}$ then $h \in H_{\tau_1}$, and if $(g,h) \in R$ and $h \in H_{\tau_1}$ then $g \in G_{\pi_1}$. Let $(g,h) \in R$ and assume $g \in G_{\pi_1}$. Since

$$\mathrm{Hom}_{G_1 \times H_1}(\omega, \pi_1 \otimes_{\mathbb{C}} \tau_1) \neq 0 \implies \mathrm{Hom}_{G_1 \times H_1}(\omega, \pi_1 \otimes_{\mathbb{C}} h\tau_1) \neq 0,$$

it follows that $\tau_1 \cong h\tau_1$ as H_1 representations, and hence as $Z'H_1$ representations. So $h \in H_{\tau_1}$. The other statement has a similar proof. Now let $v' \colon H_{\tau_1} \to \mathbb{C}^\times$ be the composition of v with the above map. Let $\widetilde{\tau_1} = v' \otimes_{\mathbb{C}} \widehat{\tau_1}$. Then if $(g,h) \in R \cap (G_{\pi_1} \cap H_{\tau_1})$ and $\varphi \in \mathcal{S}$,

$$\begin{aligned} T(\omega(g,h)\varphi) =& (\widehat{\pi_1}(g) \otimes \widehat{\tau_1}(h))T(\varphi) \\ =& (\widetilde{\pi_1}(g) \otimes \widetilde{\tau_1}(h))T(\varphi). \end{aligned}$$

Now we prove that such $\widehat{\pi_1}$ and $\widehat{\tau_1}$ exist. Let L be the subgroup of $G \times H$ generated by $R \cap (G_{\pi_1} \times H_{\tau_1})$ and $Z \times Z'$. We claim that $\pi_1 \otimes_{\mathbb{C}} \tau_1$ extends to an element of $\mathrm{Irr}(L)$ so that T is an $R \cap (G_{\pi_1} \times H_{\tau_1})$ map. To this end, we prove that $\omega(g,h)\ker(T) = \ker(T)$ for $(g,h) \in R \cap (G_{\pi_1} \times H_{\tau_1})$. Let $(g,h) \in R \cap (G_{\pi_1} \times H_{\tau_1})$. There is an isomorphism $g^{-1}\pi_1 \otimes_{\mathbb{C}} h^{-1}\tau_1 \to \pi_1 \otimes_{\mathbb{C}} \tau_1$ of $G_1 \times H_1$ representations. Let T' be the composition of the three maps:

$$\omega \to (g^{-1}, h^{-1})\omega \to g^{-1}\pi_1 \otimes_{\mathbb{C}} h^{-1}\tau_1 \to \pi_1 \otimes_{\mathbb{C}} \tau_1.$$

Here the first map is given by $\varphi \mapsto \omega(g,h)\varphi$ and the second by $\varphi \mapsto T(\varphi)$. Then T' is a $G_1 \times H_1$ map. By multiplicity preservation, there exists $c \in \mathbb{C}^\times$ such that $T' = cT$. It follows that $T(\omega(g,h)\ker(T)) = 0$. Now since $\omega(g,h)\ker(T) = \ker(T)$ for $(g,h) \in R \cap (G_{\pi_1} \times H_{\tau_1})$, and since T is surjective, we can define a representation of $R \cap (G_{\pi_1} \times H_{\tau_1})$ on the space of $\pi_1 \otimes_{\mathbb{C}} \tau_1$ extending $(\pi_1 \otimes \tau_1)|_{G_1 \times H_1}$. This representation is compatible with the representation $(\pi_1 \otimes \tau_1)|_{Z \times Z'}$, so that we obtain an extension of $\pi_1 \otimes \tau_1$ to an element of $\mathrm{Irr}(L)$. Now L is an open normal subgroup of $G_{\pi_1} \times H_{\tau_1}$ and $(G_{\pi_1} \times H_{\tau_1})/L$ is a finite abelian group. By Lemma 2.3 of [GK], there is an element $\widehat{\pi_1} \otimes_{\mathbb{C}} \widehat{\tau_1}$ of $\mathrm{Irr}(G_{\pi_1} \times H_{\tau_1})$ whose restriction to L contains our extension of $\pi_1 \otimes_{\mathbb{C}} \tau_1$ to L. Since in fact $\widehat{\pi_1}$ and $\widehat{\tau_1}$ are extensions of π_1 and τ_1, the proof is complete. $\square$

We come now to the main theorem. We remind the reader that if G, H, etc., are as in the last section, then the following theorem requires that the residual characteristic of k is odd.

Theorem 4.4. *Let $\pi \in \mathrm{Irr}_0(G)$.*

(1) *(Howe duality) Assume that some constituent π_1 of $\pi|_{G_1}$ lies in $\mathcal{R}(G_1)$. Then there exists $\tau \in \mathrm{Irr}_0(H)$ such that*

$$\mathrm{Hom}_R(\omega, \pi \otimes_{\mathbb{C}} \tau) \neq 0;$$

(2) *(Howe duality) Suppose that $\tau, \tau' \in \mathrm{Irr}(H)$. Then*

$$\mathrm{Hom}_R(\omega, \pi \otimes_{\mathbb{C}} \tau) \neq 0, \quad \mathrm{Hom}_R(\omega, \pi \otimes_{\mathbb{C}} \tau') \neq 0 \implies \tau \cong \tau';$$

(3) *(Multiplicity preservation) Suppose that $\tau \in \mathrm{Irr}(H)$. Then*

$$\dim_{\mathbb{C}} \mathrm{Hom}_R(\omega, \pi \otimes_{\mathbb{C}} \tau) \leq 1.$$

Moreover, the analogous results hold if the roles of G and H are interchanged.

Proof of (1). We begin with some definitions. Let V be the space of π, and let $V_1 \subset V$ be the space of π_1. Since $\pi|_{G_1}$ is multiplicity free, G_{π_1} is the group of $g \in G$ such that $\pi(g)V_1 = V_1$. Let $\widetilde{\pi_1}$ be the extension of π_1 to G_{π_1} defined by $\widetilde{\pi_1}(g) = \pi(g)$ for $g \in G_{\pi_1}$. Let $\tau_1 = \theta(\pi_1) \in \mathrm{Irr}(H_1)$, and let $T\colon \omega \to \pi_1 \otimes_{\mathbb{C}} \tau_1$ be a nonzero $G_1 \times H_1$ map. Let W_1 be the space of τ_1. Finally, let α be the central character of π.

We first define an extension of τ_1 to a representation of $Z'H_1$ so that T is an $R \cap (ZG_1 \times Z'H_1)$ map. Let $\beta = (\alpha^{-1}\chi) \circ \iota^{-1}$. Define $\tau_1(zh_1) = \beta(z)\tau_1(h_1)$ for $z \in Z'$ and $h_1 \in H_1$. To see that this defines an extension of τ_1 to an element of $\mathrm{Irr}(Z'H_1)$ it suffices to show that if $z \in Z' \cap H_1$ then $\tau_1(z) = \beta(z)$. Let $z \in Z' \cap H_1$ and $\varphi \in \mathcal{S}$. Then $(\iota^{-1}(z), z) \in G_1 \times H_1$ and

$$\begin{aligned} T(\omega(\iota^{-1}(z), z)\varphi) =& (\chi \circ \iota^{-1})(z) T(\varphi) \\ (1 \otimes \tau_1(z)) T(\varphi) =& \beta(z) T(\varphi). \end{aligned}$$

Since T is surjective, $\tau_1(z) = \beta(z)$. For the remainder of the proof, we regard τ_1 as a representation of $Z'H_1$. Then T is evidently an $R \cap (ZG_1 \times Z'H_1)$ map.

Now let $\widetilde{\tau_1}$ be as in Lemma 4.3, and let τ be an element of $\mathrm{Irr}(H)$ such that $\widetilde{\tau_1}$ is a constituent of $\tau|_{H_{\tau_1}}$. Such a τ exists by Lemma 2.3 of [GK]. We will define a nonzero R map from ω to $\pi \otimes_{\mathbb{C}} \tau$. Let A be the finite set $R/(R \cap (G_{\pi_1} \times H_{\tau_1}))$. There is an action of R on A. We will denote by $\overline{(g,h)}$ the element of A determined by $(g,h) \in R$. For $a = \overline{(g,h)} \in A$, define $T_a\colon \omega \to \pi \otimes_{\mathbb{C}} \tau$ by

$$T_a = (\pi(g) \otimes \tau(h)) \circ T \circ \omega(g^{-1}, h^{-1}).$$

Since T is an $R \cap (G_{\pi_1} \times H_{\tau_1})$ map, the definition of T_a does not depend on the choice of representative for a. Note that if $(g,h) \in R$ and $a \in A$ then

$$T_a \circ \omega(g,h) = (\pi(g) \otimes \tau(h)) \circ T_{(g^{-1},h^{-1})a}.$$

Define $T' : \omega \to \pi \otimes_{\mathbb{C}} \tau$ by

$$T' = \sum_{a \in A} T_a.$$

Then T' is an R map. Finally, $T' \neq 0$. To see this, note first that if $a = \overline{(g,h)}$, then the image of T_a is $\pi(g)V_1 \otimes \tau(h)W_1$. It follows that $\mathrm{im}(T_a) \cong \mathrm{im}(T_{a'})$ as $G_1 \times H_1$ representations if and only if $a = a'$. Hence, the $\mathbb{C}$ subspace spanned by the $\mathrm{im}(T_a)$ for $a \in A$ is the direct sum of these subspaces, and $T' \neq 0$.

Proof of (2). Assume the nonvanishing of the homomorphism spaces. Let β and β' be the central characters of τ and τ', respectively. We first prove that $\beta|_{Z'} = \beta'|_{Z'}$. Let $T : \omega \to \pi \otimes_{\mathbb{C}} \tau$ be a nonzero R map. Let $\varphi \in \mathcal{S}$ be such that $T(\varphi) \neq 0$. Let $z \in Z$. Then

$$\begin{gathered} T(\omega(z, \iota(z))\varphi) = (\pi(z) \otimes \tau(\iota(z)))T(\varphi) \\ \chi(z)T(\varphi) = \alpha(z)\beta(\iota(z))T(\varphi). \end{gathered}$$

Hence, $\chi|_Z = \alpha \cdot \beta \circ \iota|_Z$. Similarly, $\chi|_Z = \alpha \cdot \beta' \circ \iota|_Z$. Thus, $\beta|_{Z'} = \beta'|_{Z'}$.

By Lemma 4.2 and its proof, since $\beta|_{Z'} = \beta'|_{Z'}$, $\tau|_{Z'H_1}$ and $\tau'|_{Z'H_1}$ are multiplicity free and equivalent. By Lemma 2.4 of [GK], there is a character v of H trivial on $Z'H_1$ such that $\tau' \cong \nu \otimes_{\mathbb{C}} \tau$. We claim that $\nu = 1$. We will use the notation of Lemma 4.2 and its proof. Since $\tau' \cong \nu \otimes_{\mathbb{C}} \tau$, there exist nonzero maps

$$T : \mathcal{S} \to V \otimes_{\mathbb{C}} W, \qquad T' : \mathcal{S} \to V \otimes_{\mathbb{C}} W$$

such that T is an R map, T' is a $\mathbb{C}$ linear map satisfying

$$T'(\omega(g,h)\varphi) = \nu(h)(\pi(g) \otimes \tau(h))T'(\varphi)$$

for $\varphi \in \mathcal{S}$, $(g,h) \in R$, and the compositions

$$\mathcal{S} \xrightarrow{T} V \otimes_{\mathbb{C}} W \to V_1 \otimes_{\mathbb{C}} W_1, \qquad \mathcal{S}' \xrightarrow{T'} V \otimes_{\mathbb{C}} W \to V_1 \otimes_{\mathbb{C}} W_1$$

are nonzero, where the last maps are projection. Let t and t' denote these last nonzero $G_1 \times H_1$ maps, respectively. By multiplicity preservation, there exists nonzero $c \in \mathbb{C}$ such that $t' = ct$. By the proof of Lemma 2.4 of [GK] it suffices to show that $\nu(H_{\tau_1}) = 1$. Let $h \in H_{\tau_1}$. Since $\tau|_{H_1}$ is multiplicity free, $\tau(h)W_1 = W_1$. Let $g \in G$ be such that $(g,h) \in R$. Then $\pi(g)V_1 = V_1$. Let $\varphi \in \mathcal{S}$ be such that $t'(\varphi) = ct(\varphi) \neq 0$. Then

$$\begin{aligned} t'(\omega(g,h)\varphi) &= \nu(h)(\pi(g) \otimes \tau(h))t'(\varphi) \\ c(\pi(g) \otimes \tau(h))t(\varphi) &= c\nu(h)(\pi(g) \otimes \tau(h))t(\varphi). \end{aligned}$$

Hence, $\nu(h) = 1$.

Proof of (3). Again we will use the notation of Lemma 4.2 and its proof. We may assume that

$$\dim_{\mathbb{C}} \mathrm{Hom}_R(\omega, \pi \otimes_{\mathbb{C}} \tau) \neq 0.$$

By Lemma 4.2, $\tau|_{H_1}$ is also multiplicity free. Let $T_1, T_2 \in \mathrm{Hom}_R(\omega, \pi \otimes_{\mathbb{C}} \tau)$. For each i between 1 and M , let

$$p_i : V \otimes_{\mathbb{C}} W \to V_i \otimes_{\mathbb{C}} W_{j(i)}$$

be the projection map. Since

$$\dim_{\mathbb{C}} \mathrm{Hom}_{G_1 \times H_1}(\omega, \pi_1 \otimes_{\mathbb{C}} \tau_1) = 1,$$

it follows that there exists $c \in \mathbb{C}$ such that $p_1 \circ T_2 = cp_1 \circ T_1$. Fix i between 1 and M. As in the proof of Lemma 4.2, there exists $(g, h) \in R$ such that $\pi(g)V_1 = V_i$, $\tau(h)W_1 = W_{j(i)}$ and

$$p_i \circ T_1 = (\pi(g) \otimes \tau(h)) \circ p_1 \circ T_1 \circ \omega(g^{-1}, h^{-1}).$$

It follows that $p_i \circ T_2 = cp_i \circ T_1$ for all i. Since the images of T_1 and T_2 lie in the direct sum of the $V_i \otimes_{\mathbb{C}} W_{j(i)}$ for i between 1 and M, it follows that $T_2 = cT_1$. □

5. Frobenius reciprocity and Strong Howe duality for Ω^+. We now show that strong Howe duality holds for Ω^+ with respect to $\mathrm{Irr}_0(G)$ and $\mathrm{Irr}_0(H)$. Via Frobenius reciprocity, this is a consequence of the main result of the last section.

Lemma 5.1. *If $\pi \in \mathrm{Irr}(G)$ and $\tau \in \mathrm{Irr}(H)$ then there is a $\mathbb{C}$-isomorphism*

$$\mathrm{Hom}_R(\omega, \pi \otimes_{\mathbb{C}} \tau) \cong \mathrm{Hom}_{G \times H}(\Omega^+, \pi \otimes_{\mathbb{C}} \tau).$$

Proof. Since $\Delta_R = 1$, $\Delta_{G \times H} = 1$, and $(\pi \otimes_{\mathbb{C}} \tau)^\vee|_R{}^\vee = \pi \otimes_{\mathbb{C}} \tau$, by Frobenius reciprocity as on page 24 of [BZ],

$$\mathrm{Hom}_R(\omega, \pi \otimes_{\mathbb{C}} \tau) \cong \mathrm{Hom}_{G \times H}(\text{c-Ind}_R^{G \times H} \omega, \pi \otimes_{\mathbb{C}} \tau).$$

Since $\Omega^+ \cong \text{c-Ind}_R^{G \times H} \omega$, the lemma follows. □

Theorem 5.2. *If the residual characteristic of k is odd, then strong Howe duality holds for Ω^+ with respect to $\mathrm{Irr}_0(G)$ and $\mathrm{Irr}_0(H)$.*

Proof. By Proposition 1.1 and Theorem 4.4, it suffices to show that $\mathcal{R}_0(G)$ and $\mathcal{R}_0(H)$ as defined in the last section are $\mathcal{R}(G) \cap \mathrm{Irr}_0(G)$ and $\mathcal{R}(H) \cap \mathrm{Irr}_0(H)$, respectively. By Theorem 4.4 and Lemma 5.1, $\mathcal{R}_0(G) \subset \mathcal{R}(G) \cap \mathrm{Irr}_0(G)$ and $\mathcal{R}_0(H) \subset \mathcal{R}(H) \cap \mathrm{Irr}_0(H)$. Let $\pi \in \mathcal{R}(G) \cap \mathrm{Irr}_0(G)$. Let $\text{c-Ind}_R^{G \times H} \omega \to \pi$ be a nonzero G map. There exists a closed subgroup $Z'' \subset Z'$ such that $R \cap (1 \times Z'') = 1$ and $(G \times H)/R(1 \times Z'')$ is finite. It follows that there is a G_1 map $\text{c-Ind}_R^{R(1 \times Z'')} \omega \to \pi$ and an R map $\omega \to \text{c-Ind}_R^{R(1 \times Z'')} \omega$ such that the composition

$$\omega \to \text{c-Ind}_R^{R(1 \times Z'')} \omega \to \pi$$

is nonzero. Hence, $\pi \in \mathcal{R}_0(G)$. The proof of the remaining statement is analogous. □

6. The problem of Howe duality for Ω. In this last section we consider Howe duality for Ω when $G \neq G'$, m is even and the residual characteristic of k is odd. In Lemma 6.1, using Proposition 1.2, we give a condition equivalent to Howe duality for Ω. Using the condition, we prove that Howe duality for Ω does not hold in the stable range $m \geq 4n+2$. Since the condition is being currently investigated, we can also show that, conjecturally, strong Howe duality for Ω holds for $m \leq 2n$.

For the remainder of this section we will assume that m is even and the Witt index of X is $(m-2)/2$ so that $[G' : G] = 2$. We need some additional notation. Let K and a be as at the end of section 3. Let a' be a representative for the other coset of $k^\times / \mathrm{N}_k^K(K^\times)$ besides $a\,\mathrm{N}_k^K(K^\times)$. Let $X_2 = X_2' = K$. Define symmetric bilinear forms on X_2 and X_2' by $(x,y) = a\,\mathrm{T}_k^K(x\overline{y})$ and $(x,y)' = a'\,\mathrm{T}_k^K(x\overline{y})$. Then $(X_2, (\ ,\ ))$ and $(X_2', (\ ,\ )')$ have the same determinant but opposite Hasse invariant. As in section 3, $(X, (\ ,\ ))$ is the orthogonal direct sum of $(X_2, (\ ,\ ))$ and $(m-2)/2$ copies of the hyperbolic plane. Let $(X', (\ ,\ )')$ be the orthogonal direct sum of $(X_2', (\ ,\ )')$ and $(m-2)/2$ copies of the hyperbolic plane. Note that G is also the group of $g \in G'$ such that there exists $h \in \mathrm{GO}(X')$ such that there $\lambda(h) = \lambda(g)$. To avoid confusion, we will write $\mathcal{R}_0(G,X)$ for $\mathcal{R}_0(G)$ defined with respect to X; $\mathcal{R}_0(G,X')$, $\mathcal{R}(G_1,X)$ and $\mathcal{R}(G_1,X')$ are similarly defined. Finally, we remark that if the residual characteristic of k is odd then the subscript 0 is unnecessary since in this case $[G : k^\times G_1] = [H : k^\times H_1] = [\mathrm{N}_k^K(K^\times) : k^{\times 2}] = 2$. See [GK].

Lemma 6.1. *Assume the residual characteristic of k is odd. Then*

1. *If $\mathcal{R}(G_1,X) \cap \mathcal{R}(G_1,X') = \emptyset$ then strong Howe duality holds for Ω;*
2. *If $\mathcal{R}(G_1,X) \cap \mathcal{R}(G_1,X') \neq \emptyset$ then Howe duality for Ω does not hold.*

Proof. Let g be a representative for the nontrivial coset of G'/G. By Lemma 4.2, part (1) of Theorem 4.4 and Lemma 2.3 of [GK], $\mathcal{R}(G,X) \cap g\mathcal{R}(G,X) = \emptyset$ if and only if $\mathcal{R}(G_1,X) \cap g\mathcal{R}(G_1,X) = \emptyset$; note that $[G : k^\times G_1] = 2$. Using the Schrödinger model for r, one can verify that $(g,1)\cdot\omega \cong \omega'$ as representations of G_1. It follows that $g\mathcal{R}(G_1,X) = \mathcal{R}(G_1,X')$. Applying Proposition 1.2 and Theorem 5.2 now gives the statements. $\square$

If $m \geq 4n+2$ and the residual characteristic of k is odd, then Howe duality does not hold for Ω. If $m \geq 4n+2$, i.e., if the Witt indices of X and X' are at least $2n$, so that X and Y and X' and Y lie in the stable range, then by [MVW], $\mathrm{Irr}(G_1) = \mathcal{R}(G_1,X) = \mathcal{R}(G_1,X')$. Thus, in this case, if the residual characteristic of k is odd, then by Lemma 6.1 Howe duality does not hold for Ω.

If $m \leq 2n$ and the residual characteristic of k is odd, then Strong Howe duality for Ω can be expected to hold. It is conjectured that if $m \leq 2n$ then $\mathcal{R}(G_1,X) \cap \mathcal{R}(G_1,X') = \emptyset$. By the lemma, if this conjecture is true and if the residual characteristic of k is odd, then Strong Howe duality for Ω holds. This conjecture, called the theta dichotomy, has been partly proven in the case of unitary groups in [HKS]. See also [KR].

If $2n < m < 4n+2$ then present results or conjectures do not seem to determine whether $\mathcal{R}(G_1,X) \cap \mathcal{R}(G_1,X') = \emptyset$. It is conjectured that if $m > 2n$ then $\mathcal{R}(G_1,X) \cup$

$\mathcal{R}(G_1, X') = \mathrm{Irr}(G_1)$. Again, see [HKS]. However, this conjecture is not strong enough to settle the question. Note that if for some m_0, $\mathcal{R}(G_1, X_{m_0}) \cap \mathcal{R}(G_1, X'_{m_0}) \neq \emptyset$ then $\mathcal{R}(G_1, X_m) \cap \mathcal{R}(G_1, X'_m) \neq \emptyset$ for all $m \geq m_0$. This follows from $\mathcal{R}(G_1, X_{m_0}) \subset \mathcal{R}(G_1, X_m)$ and $\mathcal{R}(G_1, X'_{m_0}) \subset \mathcal{R}(G_1, X'_m)$ for $m \geq m_0$, which is called persistence in [HKS]. Thus, if the residual characteristic of k is odd and $\mathcal{R}(G_1, X_{m_0}) \cap \mathcal{R}(G_1, X'_{m_0}) \neq 0$, then by Lemma 6.1 Howe duality does not hold for Ω for $m \geq m_0$.

Finally, we mention that Proposition 1.2 could be used to prove that strong Howe duality holds for Ω when $G \neq G'$, m is odd and the residual characteristic of k is odd. Let $\mathrm{Irr}_0(G')$ be the set of $\pi \in \mathrm{Irr}(G')$ such that the constituents of $\pi|_G$ lie in $\mathrm{Irr}_0(G)$. There is a normal series from G' to G with $\mathbb{Z}/2\mathbb{Z}$ quotients. Thus, repeated verification of the condition of Proposition 1.2 along with repeated application of Proposition 1.2 would prove strong Howe duality for Ω with respect to $\mathrm{Irr}_0(G')$ and $\mathrm{Irr}_0(H)$.

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