1 Contrasts

To illustrate the concepts in this chapter, we will use the example of the hand steadiness experiment discussed in earlier lectures. Recall that there are four groups which involve sleep deprivation of 12, 18, 24, or 30 hours, with 8 subjects per group. As before, we will refer to the treatment groups as 1, 2, 3, and 4, and the response is a hand steadiness score. Refer to the SAS program for today's lecture to see how to perform these calculations with SAS.

The term **contrast** is used to describe a comparison of means. Specifically, a contrast is a linear combination of the population means,

$$L = \sum_{i=1}^{g} w_i \mu_i \text{ that also satisfies } \sum_{i=1}^{g} w_i = 0.$$

We require that the coefficients w_i sum to zero so that the comparison is meaningful (we would not be interested in $\mu_1 - 3\mu_2$ for example). For contrasts we are generally interested in testing the null hypothesis $H_0 : L = \sum_{i=1}^{g} w_i \mu_i = 0$, against the alternative hypothesis $H_A : L = \sum_{i=1}^{g} w_i \mu_i \neq 0$. Notice that our text uses a slightly different notation to denote a contrast. Instead of the symbol L, which we use to express the value of the contrast, they use $w(\{\mu_i\})$, which focuses attention on the w_i values used to define the contrast. They also point out that the value of a contrast does not depend on the restrictions on the α_i values.

As an example, if we wished to test whether the two lowest sleep deprivation levels differed from each other, we can express the null hypothesis as $H_0 : 1\mu_1 - 1\mu_2 = \mu_1 - \mu_2 = 0$. Here w_1=1 and w_2= -1, so $w_1 + w_2 = 0$ as required. This is an example of a **pairwise contrast**, which is defined as a contrast involving only two groups. An example of a **non-pairwise contrast** would be if we wished to test if the average of the two lowest sleep deprivation groups differed from the average of the two highest sleep deprivation groups. We can express this null hypothesis as $H_0 : (\mu_1 + \mu_2)/2 - (\mu_3 + \mu_4)/2 = 0$. Here w_1=1/2, w_2= 1/2, w_3= -1/2, w_4= -1/2, so again $w_1 + w_2 + w_3 + w_4 = 0$, as required by the definition of a contrast. If the lowest sleep deprivation level is considered a control group, what would be the contrast comparing the control group to the average of the other groups? Another type of contrast is a **polynomial contrast**, which is a comparison among the levels of a quantitative factor (like a dose level) that correspond to a particular polynomial shape for the response. Contrasts for a linear trend or a quadratic trend are the two most commonly used polynomial contrasts.

A property of a set of contrasts, called orthogonality, is useful when considering sets of tests. Two contrasts

$$L_1 = \sum_{i=1}^{g} w_{1i} \mu_i \text{ and } L_2 = \sum_{i=1}^{g} w_{2i} \mu_i \text{ are } \textbf{orthogonal} \text{ if } \sum_{i=1}^{g} \frac{w_{1i} w_{2i}}{n_i} = 0.$$

If the group sample sizes are equal then this is equivalent to $\sum w_{1i} w_{2i} = 0$. In the examples above, if we identify the first contrast as L_1 and the second contrast as L_2, then w_{11}=1, w_{12}= -1, $w_{13} = w_{14} = 0$, are the coefficients for L_1 and w_{21}=1/2, w_{22}= 1/2, w_{23}= -1/2, w_{24}= -1/2, are the coefficients for L_2. Then if the sample sizes are equal,

$$\sum_{i=1}^{g} w_{1i} w_{2i} = (1)(1/2) + (-1)(1/2) + (0)(-1/2) + (0)(-1/2) = 0,$$

so L_1 and L_2 are orthogonal. Orthogonal contrasts are statistically independent, so that the outcome of testing one contrast is independent of the outcome of testing the other contrast. In our example, whether or not the first two sleep deprivation groups differ from each other (in terms of hand steadiness) gives no information about whether the average of the first two groups differ from the average of the other two groups. A set of more than two contrasts is **mutually orthogonal** if each pair of contrasts in the set is orthogonal to each other. The concept of a contrast or a set of contrasts at first seems somewhat esoteric, but in fact it is essential to understand these concepts to fully understand ANOVA, particularly in complicated situations. The contrasts that you will use should depend on the questions of scientific interest from your experiment.

1.1 Inference for contrasts

We can estimate the contrast

$$L = \sum_{i=1}^{g} w_i \mu_i \text{ with } \widehat{L} = \sum_{i=1}^{g} w_i \overline{y}_{i\cdot} \text{ and } \widehat{Var}(\widehat{L}) = \ MS_E \ \sum_{i=1}^{g} \frac{w_i^2}{n_i},$$

which leads to a t test of $H_0 : L = \sum_{i=1}^{g} w_i \mu_i = \delta$,

$$t = \frac{\widehat{L} - \delta}{s.e.(\widehat{L})} = \frac{\sum_{i=1}^{g} w_i \overline{y}_{i\cdot} - \delta}{\sqrt{\widehat{Var}(\widehat{L})}}.$$

For a two-tailed test, the t value is compared to $t_{\alpha/2,df}$, where *df* is the degrees of freedom for MS_E ($df = N - g$ for 1 way ANOVA). Confidence intervals for L can also be constructed as $\widehat{L} \pm \ t_{\alpha/2,df} \ s.e.(\widehat{L})$. Alternatively, we can compute a sum of squares for the contrast L:

$$SS_L = SS_w = \frac{(\sum_{i=1}^{g} w_i \overline{y}_{i\cdot})^2}{\sum_{i=1}^{g} \frac{w_i^2}{n_i}}.$$

The sum of squares has 1 degree of freedom, so a test of $H_0 : L = \sum_{i=1}^{g} w_i \mu_i = 0$ is conducted by comparing $F = \frac{SS_L/1}{MS_E}$,which is compared to an F distribution with numerator degrees of freedom = 1 and denominator degrees of freedom = df for MS_E, which is $N - g$ for 1 way ANOVA.

1.2 Orthogonal contrasts form a partition of SS_{Trt}

As noted in the text, we can form as many orthogonal contrasts as we have degrees of freedom between groups. Essentially these orthogonal contrasts partition the SS_{Trt} into SS_{L_i} terms that allow us to separate the total between group sum of squares into parts attributable to different contrasts. This can be a powerful tool for understanding treatment effects. One example of this is when we use orthogonal polynomial contrasts to partition dose effects into parts attributable to linear trend, quadratic trend, and higher-order trends. For equally-spaced dosage levels with equal-sample-size groups, the coefficients for orthogonal polynomial contrasts are given in Table D.6 on page 630 of our text.