1 Chapter 5: Fixed versus Random effects models

So far all of the treatment effects that we have studied were fixed, meaning that the observed levels of the treatment factor are the only levels of interest. In this model,

$$y_{ij} = \mu + \tau_i + e_{ij},$$

the t values of τ_i, $\tau_1, \tau_2, ..., \tau_t$, are the only effects of interest, and e_{ij} is the only random term, e_{ij} has a normal distribution with mean 0 and variance σ_e^2 ($e_{ij} \sim N(0, \sigma_e^2)$). However, sometimes the treatment levels under study are only a sample of the possible treatment levels, and we wish to generalize our results to this larger set of levels. The high temperature alloy casting example is like this, where the three fabrication castings in the study are only a sample of the set of fabrication castings. For this type of study a more appropriate model is the random effects model:

$$y_{ij} = \mu + a_i + e_{ij},$$

Here the a_i terms are now random, $a_i \sim N(0\ ,\ \sigma_a^2\ )$, the e_{ij} terms have the same assumptions as before, and the a_i and e_{ij} are independent. Now the a_i 's are viewed as a random sample from a population of a_i ‘s, and the ANOVA H_0 is H_0: $\sigma_a^2 = 0$ vs. $H_a : \sigma_a^2 > 0$.

How do you know if an effect is fixed or random?

1. How were the levels for the factor in question chosen?
2. Is it desired to generalize the results to levels that weren't used?

Implications for random effects

1. Generally you are not interested in doing multiple comparisons. There may be interest in estimating σ_a^2 , and σ_e^2 , or functions of them, like ratios.
2. The denominator of the F statistic may change (this occurs in more complex designs).

Analysis of random effects models

The null hypothesis, H_0: $\sigma_a^2 = 0$ vs. $H_a : \sigma_a^2 > 0$ can be tested by the ANOVA F statistic just as with the fixed effects model, as can be seen from an examination of the expected mean squares. The text obtains estimators of the variance components by equating observed mean squares to their expected values, then solving for the variance components. This method, often called the method of moments, can encounter problems such as negative variance component estimates. Instead, we will use REML (restricted maximum likelihood, similar to maximum likelihood) estimators that are computed using Proc MIXED in SAS.

1.1 The Intraclass Correlation Coefficient

One measure of interest in a random effects model is the ratio of the variance of the random effect to the total variance of the response. This quantity is called the intraclass correlation coefficient:

$$\rho_I = \frac{\sigma_a^2}{\sigma_a^2 + \sigma_e^2}$$

As mentioned in the text, ρ_I is a measure of similarity of the observations within the same treatment group level. The text shows how to calculate estimates of the coefficient using method of moments estimators, but we will use a maximum-likelihood-based estimator, as shown in the code.

1.2 Power and sample size for the random effects model

For the completely randomized design with random effects and one-way treatment structure, when the null hypothesis is true, the F statistic has an F distribution with $t-1$ and $t(r-1)$ degrees of freedom. When the null hypothesis is false, the distribution of the F statistic is a multiple of a central F distribution with $t-1$ and $t(r-1)$ degrees of freedom, where the multiplicative constant is equal to:

$$\lambda^2 = 1 + \frac{r\sigma_a^2}{\sigma_e^2},$$

so that we can calculate the power by

$$P\left[F > \frac{1}{\lambda^2} F_{\alpha,\nu_1,\nu_2}\right],$$

where λ^2 is the value of the multiplicative constant under H_a and F_{α,ν_1,ν_2} is a tabled F value for $\alpha =$ Type I error, and degrees of freedom ν_1 and ν_2. Power can then be calculated by specifying a value for the ratio σ_a^2/σ_e^2, or via the way presented in the text where a percentage P increase in σ_y caused by σ_a^2 is specified.

1.3 Random subsamples

Sometimes subsamples are collected from an experimental unit and measured separately. We need to use a model that reflects the distinction between variation between experimental units and between subsamples within an experimental unit. The model is:

$$y_{ij} = \mu + \tau_i + e_{ij} + d_{ijk},$$

where d_{ijk} is the random effect of the k^{th} subsample within the j^{th} experimental unit in the i^{th} treatment, and the other terms are as defined in previous models. The text shows the expected value of the mean squares of these terms, which are used to determine the form of the F test. The use of expected mean squares to determine how to form the appropriate F statistics to test hypotheses of interest will become increasingly important.

1.4 Random subsamples with unequal sample sizes

At the end of the chapter, the random subsample model is considered again, but now with unequal replicate and subsample numbers. The table of expected mean squares is shown on page 167:

Source of variation	Expected mean squares
Treatments	$\sigma_d^2 + c_1\sigma_e^2 + c_2\sigma_a^2$
Error	$\sigma_d^2 + c_3\sigma_e^2$
Sampling	σ_d^2

Unless $c_1 = c_3$, the ratio MST/MSE does not have the proper behavior under the null hypothesis. As shown in the text, in these cases we can construct a linear function of mean squares:

$$M = a_1 MSE + a_2 MST$$

that will have expected value $\sigma_d^2 + c_1\sigma_e^2$ so that an appropriate F ratio can be formed. For these linear functions of mean squares:

$$M = a_1 MS_1 + a_2 MS_2 + \cdots + a_k MS_k$$

it was shown by Satterthwaite (1946) that the distribution of this linear function M can be approximated by a chi-square distribution with degrees of freedom equal to:

$$\nu = \frac{M^2}{\sum_{i=1}^{k} \frac{(a_i MS_i)^2}{\nu_i}}$$